General Relativity and Cosmology II

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Based on prof. Victor Gorbenko's lectures in Spring 2024

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Contents

1 Introduction

Introduction

Throughout your studies, you may have encountered some of the following statements about the universe:

- The universe is homogeneous and isotropic.
- It is currently expanding at an accelerated rate.
- The universe was once extremely hot and dense.
- Relic black-body radiation permeates the universe.
- There are two main types of matter: ordinary matter and dark matter.
- The universe is dominated by dark energy.
- Light elements such as deuterium, helium-3, helium-4, and lithium were formed in the first few minutes after the Big Bang.
- The universe underwent a rapid phase of inflation in its early stages.
- The large-scale structures we observe today, such as clusters of galaxies, originated from quantum fluctuations.

The goal of these lectures is to delve into these statements and provide a comprehensive understanding using the principles of general relativity, equilibrium and non-equilibrium statistical mechanics, and particle physics.

Structure of the Lecture Notes

The lecture notes are organized as follows:

- **Friedmann-Robertson-Walker (FRW) Metric**: Homogeneous universe, distances and horizons, cosmological epochs, and the ΛCDM model.
- **Thermal History**: Evolution of matter and radiation, Cosmic Microwave Background (CMB), Big Bang Nucleosynthesis (BBN), with a brief discussion on baryogenesis.
- **Dark Matter**: Evidence for dark matter and the most popular candidates.
- **Theory of Cosmological Perturbations**: CMB and the growth of structure.
- **Inflation**: The theory and implications of cosmic inflation.

Suggested References

For further reading and deeper understanding, the following references are recommended as supplementary material:

- Landau-Lifshitz, *The Classical Theory of Fields* (Volume II)
- Kolb and Turner, *The Early Universe*
- Weinberg, *Cosmology* (2007)
- Gorbunov and Rubakov, *Introduction to the Theory of the Early Universe* (especially for the second part of the course)

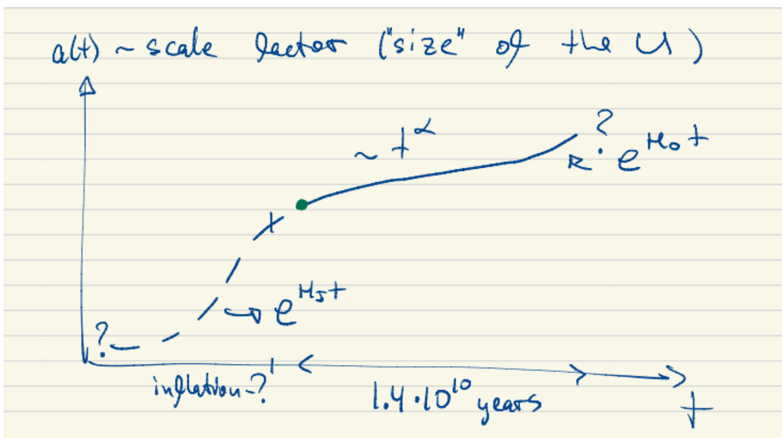
2 Expanding Universe

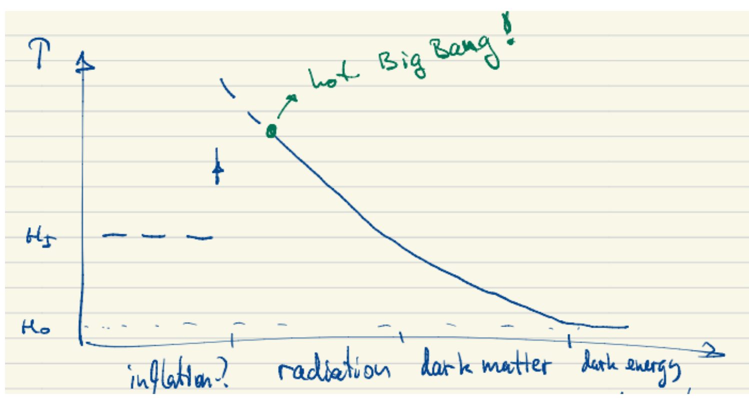
2.1 Brief history

In order to better understand the scales at which we are going to work with, we state some basic facts about the universe :

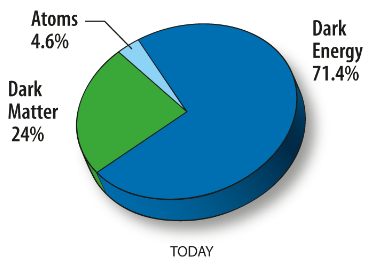
- On very large scales (on average) the universe is isotropic and homogeneous and specially flat.
- The universe has been expanding, meaning that the distances between different objects are growing.

To illustrate this point, we can draw a diagram for the scale factor (rigorously defined later) which gives a sense of the dynamics of the distances between different objects :





We can also calculate the abundance of the different components of the universe, which turn out to be:



To clarify the first point, we need to specify what we mean by large scales as in ordinary day to day experiences the universe is definitely neither homogeneous nor isotropic.

At the moment, the size of the observable universe is around $5 \times 10^3 Mpc$, and from observations, it is homogeneous on scales greater than $10Mpc$.

Remark. *It is good to have the following scales in mind :*

- *Distance between sun and earth :* $1.5 \times 10^{11} m$
- *Galaxy :* $5 \times 10^4 l.y.$
- $1pc \approx 3l.y. = 3 \times 10^{16}$

2.2 FLRW spacetime

As we stated previously, the universe is homogeneous and isotropic on large scales (that we can see). Based on these symmetries we can determine the spacial metric g_{ij} which describes the geometry at a given time (for any timeslice). We have

$$\begin{aligned} R^{(3)} &= const. \\ R_{ij} &= \frac{1}{3} R g_{ij} \\ R_{ijkl} &= cR[g_{ik}g_{jl} - g_{il}g_{jk}]. \end{aligned} \tag{2.1}$$

By taking the trace of the Riemann tensor we see that $c = \frac{1}{6}$.

Depending on the sign of the Ricci scalar, there are three different cases in general :

- $R = 0$ is just the three-plane $\mathbb{R}^3$
- $R > 0$ is the 3-sphere $\mathbb{S}^3$ defined by the equation :

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = a^2. \tag{2.2}$$

  Clearly this equation is homogeneous and isotropic. Let us use the following coordinates on the sphere :

$$\begin{aligned} x_1 &= r \sin\theta \cos\phi \\ x_2 &= r \sin\theta \sin\phi \\ x_3 &= r \cos\theta \\ x_4 &= \sqrt{a^2 - r^2} \end{aligned} \tag{2.3}$$

Taking the differentials and adding them together we have :

$$dx_1^2 + dx_2^2 + dx_3^2 = dr^2 + r^2 d\Omega_2^2 \tag{2.4}$$

where $d\Omega_2^2$ is the metric on S^2. Finally, we have :

$$dx_4^2 = \frac{r^2 dr^2}{a^2 - r^2}. \tag{2.5}$$

Putting all the ingredients together we have :

$$ds^2 = \frac{a^2 dr^2}{a^2 - r^2} + r^2 d\Omega_2^2. \tag{2.6}$$

The curvature of the sphere is proportional to a^{-2} : $R \sim a^{-2}$.

- $R < 0$ is the Hyperbolic space $\mathbb{H}^3$, also known as the Euclidean Anti-de Sitter space. Consequently, using the result for the sphere, we just take $a^2 \to -a^2$ to get a negatively curved space :

$$ds^2 = \frac{a^2 dr^2}{a^2 + r^2} + r^2 d\Omega_2^2. \tag{2.7}$$

In order to have an embedding of the hyperbolic space, we could use the following condition :

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = -a^2. \tag{2.8}$$

There is a unified description of all the metrics :

$$ds^2 = a^2 (\frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2 d\Omega_2^2), \tag{2.9}$$

where $k = 1, 0, -1$ corresponds to spherical, flat and hyperbolic space respectively.

Now we go back to describing the full 4-dimensional geometry by just adding the time coordinate :

$$ds^2 = -dt^2 + a^2(t)(\frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2 d\Omega_2^2), \tag{2.10}$$

where so far, $a(t)$ is a function of time called the scale factor. This is the metric of **Friedmann-Lemaitre-Robertson-Walker**. It has the same symmetry properties for a generic $a(t)$.

Remark. *It is possible to consider spaces with the same local metric but non-trivial topologies.*

Friedmann Equations : On large (observable) scales, the universe is well approximated by the FRW metric. In the first part of the course we will study this homogeneous approximation. Our next goal will be to determine the function $a(t)$ from the Einstein field equations.

Let us start by reminding the Einstein field equations :

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.11}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor. Sometimes it is convenient to single out the cosmological constant from $T_{\mu\nu}$:

$$T_{\mu\nu} = T^m_{\mu\nu} + \Lambda \frac{1}{8\pi G} g_{\mu\nu}. \tag{2.12}$$

Now we substitute the FRW metric to determine geometric invariants [see Carroll 8.44 [1]] :

$$\Gamma^0_{ij} = \frac{\dot{a}}{a} g_{ij}, \quad \Gamma^i_{0j} = \frac{\dot{a}}{a} \delta_{ij}, \quad \Gamma^i_{jk} = \frac{k}{a^2} g_{jk} x^i. \tag{2.13}$$

The Ricci tensor and Ricci scalar read :

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a}, \\ R_{ij} &= (\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2}) g_{ij}, \\ R &= 6(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}). \end{aligned} \tag{2.14}$$

Now we get to calculating the other side of the Einstein equation, the energy momentum tensor. Let us assume that the content of the universe is a perfect fluid, only characterized by its velocity u^μ. In this case, there are only two invariant tensors available for building the energy momentum tensor :

$$T_{\mu\nu} = A u_\mu u_\nu + B g_{\mu\nu}. \tag{2.15}$$

In the rest frame of the fluid in flat space, the above reduces to :

$$\begin{aligned} u^\mu &= (1, 0, 0, 0), \\ T_{\mu\nu} &= diag(\rho, p, p, p). \end{aligned} \tag{2.16}$$

Therefore, we have :

$$\begin{aligned} & B = p, \quad A = \rho + p \\ \Rightarrow \quad & T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} \end{aligned} \tag{2.17}$$

Now the (00) and (ij) components of the Einstein equations read :

$$\begin{aligned} (00) \Rightarrow \quad & \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3} \rho, \\ (ij) \Rightarrow \quad & 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \Lambda = -8\pi G p, \end{aligned} \tag{2.18}$$

where the first equation is called the first Friedmann equation.

Note that the Einstein equations gave us two independent equations, however, we have three unknowns. The reason is that the Einstein equations are not supposed to tell us what kind of matter we are dealing with. That has to do with the type of matter itself. Therefore, we have to specify the matter equation as well, which relates ρ and p. Simple models of matter produce :

$$p = \omega \rho \tag{2.19}$$

Depending on what we are dealing with, ω can take different values. For example, these are the mostly used values in cosmology :

$$\omega = \begin{cases} 0 & \text{Pressure-less dust} \\ \frac{1}{3} & \text{Relativistic matter \& radiation} \\ -1 & \text{Cosmological Constant} \end{cases} \tag{2.20}$$

Remark. *The null energy condition stipulates that for any future directing null vector field k^μ, $T_{ab}k^a k^b \geq 0$. In this context, the null energy condition translates into $\omega \geq -1$.*

Moreover, combining the first Friedmann equation with the (ij) component of the Einstein tensor, we get :

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho + 3p), \tag{2.21}$$

which is called the second Friedmann equation. This equation is particularly useful as the sign of $\ddot{a}$ is manifest.

Furthermore, computing the conservation of the energy-momentum tensor gives :

$$\nabla_\mu T^{\mu\nu} = \frac{\partial}{\partial t}(\rho a^3) + p\frac{\partial a^3}{\partial t} = 0. \tag{2.22}$$

Note that this is the first law of thermodynamics :

$$dE + pdV = 0. \tag{2.23}$$

Remark. *Bear in mind that the energy-momentum conservation equation is not independent from the Friedmann equations as it follows from Einstein's equations.*

Remark. *We write both Friedmann equations in here for reference :*

$$\begin{aligned} \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho, \\ \frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho + 3p). \end{aligned} \tag{2.24}$$

There are other ways of writing the Friedmann equation which can be useful in certain contexts. For starters, the Hubble parameter (constant) is defined as follows :

$$H \equiv \frac{\dot{a}}{a}. \tag{2.25}$$

Now the first Friedmann equation can be written as follows :

$$H^2 = \frac{8\pi G}{3}(\sum \rho_i), \tag{2.26}$$

where the sum goes over the different components such as the cosmological constant, matter, radiation, curvature, ...

Remark. *Note that we defined the curvature density as $\rho_k = -\frac{3}{8\pi G}\frac{k}{a^2}$.*

Furthermore, it is common to define the critical density as :

$$\rho_c = \frac{3H_0^2}{8\pi G}, \tag{2.27}$$

where H_0 is the Hubble parameter at the present ($t = t_0$). Therefore, the Friedmann equation for present time can be written as :

$$\rho_c = \sum \rho_{i,0}. \tag{2.28}$$

Moreover, define :

$$\begin{aligned} \Omega_i &= \frac{\rho_{i,0}}{\rho_c} \\ \Rightarrow \sum \Omega_i &= 1. \end{aligned} \tag{2.29}$$

Note that the above equation is the Friedmann equation at the present. In general we have :

$$H^2 = H_0^2[\Omega_\Lambda + \Omega_k(\frac{a_0}{a})^2 + \Omega_m(\frac{a_0}{a})^3 + \Omega_\gamma(\frac{a_0}{a})^4]. \tag{2.30}$$

One useful way of thinking about the above equation (which is by the way still the first Friedmann equation) is to multiply by a :

$$\dot{a}^2 = U(a) \sim \text{motion potential}. \tag{2.31}$$

In the following subsection, we will discuss solutions where one of the components dominates as well as global de-Sitter space ($\Lambda + k$).

2.3 Solutions of Friedmann equations

The Einstein static universe : This solution of the Friedmann equation has historical significance but it is not compatible with observations. Upon thinking about the universe, Einstein wanted to have a solution with :

$$\rho \neq 0, \quad \dot{a} = 0, \quad (\ddot{a} = 0). \tag{2.32}$$

It is important to note that these assumptions are not based on firm grounds. Only historically, they seemed natural to physicists.

Upon these assumptions, the Friedmann equations (2.24) reduce to :

$$\begin{aligned} &\frac{k}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho, \\ &\frac{k}{a^2} - \Lambda = 0. \end{aligned} \tag{2.33}$$

The above system of equations, admit the following solution :

$$a = \frac{1}{\sqrt{\Lambda}}, \quad k = 1, \quad \Lambda = 4\pi G\rho. \tag{2.34}$$

This is when the cosmological constant was first added to general relativity!

Flat matter dominated universe : In this case, we are assuming that the universe is matter dominated and therefore, $\Lambda \Rightarrow 0$. Moreover, $k = 0$ and $p = 0$ (dust like matter). In these limits, the Friedmann equations (2.24) reduce to :

$$\begin{aligned} \frac{\dot{a}^2}{a^2} &= \frac{G\rho}{3}, \\ 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} &= 0. \end{aligned} \tag{2.35}$$

Note that from the energy-momentum conservation (2.22) (from here on we may use the first law of thermodynamics with this phrase interchangeably) we have $\rho a^3 = const.$, because we are working with pressure-less matter. Therefore, define : $C = \rho a^3$. The above equations can now be solved easily :

$$\begin{aligned} &\frac{\dot{a}^2}{a^2} = \frac{GC}{3a^3} \Rightarrow \frac{da}{dt} = \sqrt{\frac{GC}{3a}} \Rightarrow da\sqrt{a} = dt\sqrt{\frac{GC}{3}} \\ &\Rightarrow \frac{2}{3}a^{3/2} = (t - t_0)\sqrt{\frac{GC}{3}}. \end{aligned} \tag{2.36}$$

Therefore, we have :

$$\begin{aligned} a &= a_0(t - t_0)^{2/3}, \\ \rho &= \frac{\rho_0}{(t - t_0)^2}. \end{aligned} \tag{2.37}$$

Remark. *Note that for the above solution, we have $\dot{a} > 0$ and $\ddot{a} < 0$.*

General equation of matter : Note that the energy-momentum conservation (2.22) can also be written as :

$$\begin{aligned} &\dot{\rho} + 3\frac{\dot{a}}{a}(p + \rho) = 0 \\ \Rightarrow &\frac{d\rho}{\rho} = -3(1 + \omega)\frac{da}{a} \\ \Rightarrow &\rho = \rho_0 a^{-3(1+\omega)} \end{aligned} \tag{2.38}$$

Moreover, in the spatially flat case without cosmological constant, from the first Friedmann equation (2.24) we have :

$$\begin{aligned} &\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho = a^{-3(1+\omega)} \\ \Rightarrow &a = a_0 t^{\frac{2}{3(1+\omega)}}. \end{aligned} \tag{2.39}$$

Remember that $\omega \geq -1$.

Remark. *Note that if there are two different types of matter in the universe, the one with smaller ω dominates at late times. Just like our own universe where dark energy ($\omega = -1$) has dominated all other forms of matter and energy.*

Remark. *Note that curvature can be thought of like the other matter components, with $\rho < 0$ and $\omega = -\frac{1}{3}$. You can see this from the first Friedmann equation (2.24) where curvature contributes a $\frac{k}{a^2}$ term.*

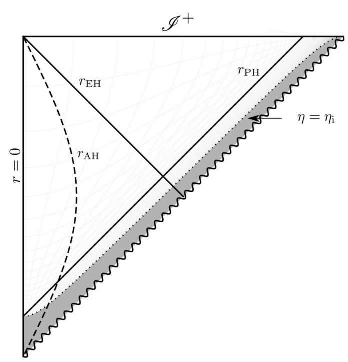
2.4 Cosmological Horizons

In this part of the discussion, we will use Penrose diagrams to have a better understanding of the FRW universe.

Define $d\eta = \frac{dt}{a(t)}$. We call η the conformal time. Using this change of coordinates, we can write the FRW metric as :

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2 = \frac{1}{\Omega^2(\eta)}(-d\eta^2 + d\vec{x}^2), \tag{2.40}$$

therefore, the FRW metric has the same causal structure as the Minkowski space, however, it consists of only part of it :



Remark. *We should check whether our change of variables to conformal time is possible. To do so, we should check if η is finite after taking the integration. It turns out that η is finite if $\omega < -\frac{1}{3}$.*

Remark. *You can see from the diagram, that there are observers who will not ever be in causal contact, ever again! The difference with the black hole horizon is that it is observer dependent. One may say that it is a democratic version :)*

As promissed in the previous part, we briefly investigate de-Sitter space. Consider the case of a space consisting only of cosmological constant.

$$\text{if}: \; k = 0, \quad \Lambda > 0, \quad \omega = -1, \quad p_m = \rho_m = 0, \tag{2.41}$$

we have :

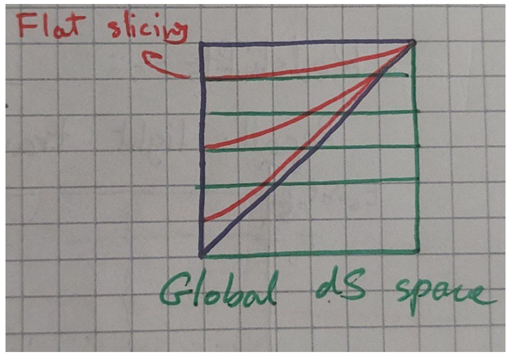
$$\rho_\Lambda = const, \quad a = e^{Ht} \tag{2.42}$$

where $H = \sqrt{\frac{\Lambda}{3}}$. In this case, however, there is no singularity at $t = -\infty$.

The global de-Sitter space can be obtained using closed slicing where we would have :

$$a = H^{-1}\cosh Ht. \tag{2.43}$$

So not only de-Sitter space is homogeneous and isotropic in space, but it is also constant in time!



2.5 Redshift and Distances

In this subsection, We would like to understand how to measure distances and time in cosmology and get some more intuition about FRW spacetimes.

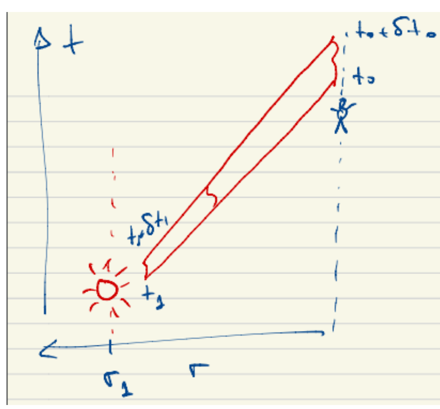
2.5.1 Redshift

We know that light travels along null geodesics. Let us suppose further that all angular parameters are constant. Furthermore, the light pulse was emitted at $r = r_1$ and $t = t_1$ and observed by us at $r = 0$ and $t = t_0$. Moving on null geodesics, we have :

$$dt = -a(t)\frac{dr}{\sqrt{1-kr^2}}$$

$$\Rightarrow \int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}} = \begin{cases} \arcsin r_1 & k>0 \\ r_1 & k=0 \\ \operatorname{arcsinh} r_1 & k<0 \end{cases} \tag{2.44}$$

Now suppose there is another light pulse sent shortly after the first one (see the figure below). The second pulse is emitted at $t = t_1 + \delta t_1$ and observed at $t = t_0 + \delta t_0$.



We then have :

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{a(t)}$$
$$\Rightarrow \frac{\delta t_0}{a(t_0)} = \frac{\delta t_1}{a(t_1)}. \tag{2.45}$$

Pay attention to the fact that we have $\delta t = \frac{1}{\omega}$. Therefore, we have :

$$\omega_0 = \omega_1 \frac{a(t_1)}{a(t_0)}. \tag{2.46}$$

Now in case we have an expanding universe, $\omega_0 < \omega_1$ or equivalently $\lambda_0 > \lambda_1$ which means that the light pulse has been **redshifted**! This gives us the idea to define a measure of redshift as :

$$z = \frac{\lambda_0 - \lambda_1}{\lambda_0} = \frac{a(t_0)}{a(t_1)} - 1. \tag{2.47}$$

Redshift is often used as a measure of time/distance, because we can measure it as we know λ's of many transitions. For example, we know the emission and absorption lines of almost all elements and we receive light from a star that if blueshifted would give a combination of some elements we know. Therefore, we can interpolate when the light was emitted.

2.5.2 Distances

In this part we want to understand how different distances are related and measured in cosmology.

First pay attention to the difference between physical distance and coordinate distance.

$$l_p = a(t) l_c$$
$$\Rightarrow \frac{dl_p}{dt} = \dot{a} l_c = \frac{\dot{a}}{a} l_p = H l_p. \tag{2.48}$$

This last equation is called Hubble's law. So far, the best measurement of the Hubble parameter is $H_0 \approx \frac{70 \frac{km}{s}}{Mpc}$. (1 Megaparsec $\approx 3 \times 10^{22} m$)

In our next step, we would like to measure distances of faraway objects. There are generally two main methods :

- Luminosity distance
- Parallax

Luminosity distance : The idea of luminosity distance comes from the fact that there are astronomical objects with known luminosity called standard candles, e.g. Cepheid variables. In the absence of expansion, apparent luminosity is :

$$P = L \frac{s}{4\pi d^2}, \tag{2.49}$$

where s is the telescope area and d being the so called luminosity distance.

Now, let us see what the luminosity distance depends on in an expanding universe :

$$P = L(\frac{S}{S_{tot}})(\frac{\hbar\omega_0}{\hbar\omega_1})(\frac{\delta t_1}{\delta t_0}), \tag{2.50}$$

with S_{tot} being the total area of the light front, the second term accounting for the redshift and the final term taking the change in time interval into account. Furthermore, we have :

$$\begin{aligned} S_{tot} &= 4\pi r^2(t_0,t_1)a^2(t_0) \\ \Rightarrow P &= L(\frac{a^2(t_1)}{a^2(t_0)}).\frac{S}{4\pi r^2(t_0,t_1)a^2(t_0)}. \end{aligned} \tag{2.51}$$

Comparing with (2.49), we have the **Luminosity distance** :

$$d = \frac{a^2(t_0)}{a(t_1)}r(t_0,t_1) = r(t_0,t_1)a(t_0)(1+z). \tag{2.52}$$

Additionally, we can express the right hand side only in terms of the redshift. Remember that from (2.44) we can write :

$$r(t_0,t_1) = \begin{cases} \sin(\int_{t_1}^{t_0}\frac{dt}{a(t)}) & k=1 \\ \int_{t_1}^{t_0}\frac{dt}{a(t)} & k=0 \\ \sinh(\int_{t_1}^{t_0}\frac{dt}{a(t)}) & k=-1 \end{cases} \tag{2.53}$$

Now we need to find $\int_{t_1}^{t_0}\frac{dt}{a(t)}$. To do so, remember the first Friedmann equation in the following form :

$$(\frac{\dot{a}}{a})^2 = H_0^2[\Omega_\Lambda + \Omega_k(\frac{a_0}{a})^2 + \Omega_m(\frac{a_0}{a})^3 + \Omega_\gamma(\frac{a_0}{a})^4] \tag{2.54}$$

Take $x = \frac{a}{a_0}$ and $A^2(x) = \Omega_\Lambda + \Omega_k\frac{1}{x^2} + \Omega_m\frac{1}{x^3} + \Omega_\gamma\frac{1}{x^4}$. Therefore we have :

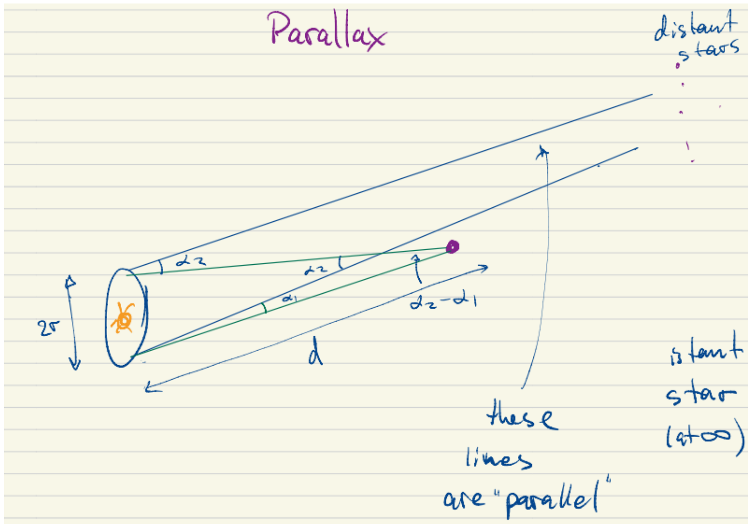
$$\begin{aligned} &\frac{1}{x}\frac{dx}{dt} = H_0A(x) \\ \Rightarrow &\int_{t_1}^{t_0}\frac{dt}{a(t)} = \int_{t_1}^{t_0}\frac{dt}{dx}\frac{dx}{a_0x} = \frac{1}{a_0H_0}\int_{\frac{1}{1+z}}^{1}\frac{dx}{x^2A(x)}. \end{aligned} \tag{2.55}$$

Finally, we can put all three cases ($k=-1,0,1$) in one equation :

$$d(z) = a_0(1+z)\frac{1}{\sqrt{-k}}\sinh(\frac{\sqrt{-k}}{a_0H_0}\int_{\frac{1}{1+z}}^{1}\frac{dx}{x^2A(x)}) \tag{2.56}$$

Remark. *Remember that the Ω_i's are constant numbers and depend only on the "matter" content of the universe. Therefore, $A(x)$ is basically a rational function of x with constant coefficients.*

Parallax



This method of measuring distances, makes use of the fact that earth is orbiting the sun. As you can see from the figure, first we assume that stars which seem not to be moving are so far away that effectively their light reaches us with parallel rays. Secondly, we have :

$$2r = d(\alpha_2 - \alpha_1), \tag{2.57}$$

where r is the radius of earth's orbit around the sun.

Remark. *This way of measuring is actually the origin of parsec as well! 1 parsec is the distance which corresponds to* $1''$ *parallax.*

Remark. *The Gaia experiment had around* $10^{-5}.1''$ *precision, which made it very useful for measuring objects at distances around* $10^{4\sim5}$ *light years away.*

Remark. *It is important to know that although the parallax method is far more accurate than the luminosity distance method, it is only useful for closer objects. In practice, however, parallax and luminosity distance (and potentially other measuring methods) are all used alongside each other to have a more accurate estimate. In this way, we can even set the coefficients of the* $A(x)$ *function to fit the data in the best possible way.*

3 Thermal History

In this section, the goal is to understand how various matter components of the universe, such as photons, electrons, protons, neutrons, neutrinos and some light atoms, exchange energy and come in and out of equilibrium. In other words, we consider the evolution of the universe as a piece-wise thermal equilibrium process. However, before getting into the details, it is best if we do a quick review of statistical thermodynamics and equilibrium physics, both to review and to set our notation.

3.1 Statistical Physics

We know that the full description of the system is given by the density matrix :

$$\hat{\rho} = \frac{1}{Z} exp(-\frac{\hat{H}}{T} + \mu_i \hat{Q}_i), \tag{3.1}$$

where Z is the partition function, $\hat{H}$ is the Hamiltonian, $\hat{Q}_i$ are representing the conserved numbers and μ_i are the chemical potentials associated with them.

In many cases, components of matter and radiation are described by free particles. However, this must not be confused so that they never interact (in that case there would never be any equilibrium at all)! It is just so that we assume that non-equilibrium processes very rapidly calm into equilibrium, after which, the particles are thought to be free, described by their distribution in momentum space.

It is convenient to use momentum space distribution in the following form :

$$\begin{aligned} N &= \frac{g}{(2\pi)^3} \int d^3p n(p), \\ \rho &= \frac{g}{(2\pi)^3} \int d^3p E(p) n(p), \end{aligned} \tag{3.2}$$

where g is the number degrees of freedom and $n(p)$ the thermal distribution function (number density). For the two different cases of Bose-Einstein or Fermi-Dirac distributions we have :

$$n_{B/F}(p) = \frac{1}{exp(\frac{E(p)-\mu}{T}) \mp 1}, \tag{3.3}$$

where :

$$E(p) = \sqrt{p^2 + m^2}. \tag{3.4}$$

Remark. *Note that we did not write the thermal distribution functions as functions of the momentum vector! This comes from the isotropy assumption that we discussed in the previous section.*

$$n(\vec{p}) = n(|\vec{p}|) = n(p). \tag{3.5}$$

Now we can consider two limits : the Relativistic ($p \gg m$) or Non-relativistic ($p \ll m$) limit.

The Relativistic Limit :

$$T \gg m_i,\ E(p) \sim |p|. \tag{3.6}$$

In this limit, both ρ and N can be computed analytically :

$$\rho_i = \frac{g_i}{(2\pi)^3}\int \frac{1}{exp(\frac{|p|}{T}) \mp 1}|p|d^3p = \begin{cases} g_i\frac{\pi^2}{30}T^4 & \text{Bosons,} \\ \frac{7}{8}g_i\frac{\pi^2}{30}T^4 & \text{Fermions,} \end{cases} \tag{3.7}$$

and similarly for N :

$$N_i = \begin{cases} \frac{\xi(3)}{\pi^2}g_iT^3 & \text{Bosons,} \\ \frac{3}{4}\frac{\xi(3)}{\pi^2}g_iT^3 & \text{Fermions,} \end{cases} \tag{3.8}$$

where $\xi(z)$ is the Riemann zeta function. Using the above, we can see that the energy per particle for bosons and fermions is :

$$< E >= \frac{\rho_i}{N_i} \sim \begin{cases} 2.7T & \text{Bosons} \\ 3.15T & \text{Fermions} \end{cases} \tag{3.9}$$

It is common to define the "effective number of relativistic species" as :

$$g_* = \sum_{bosons} g_i + \frac{7}{8}\sum_{fermions} g_i, \tag{3.10}$$

after which we can write for the total energy density :

$$\rho = \frac{\pi^2}{30}g_*T^4. \tag{3.11}$$

We can also use the first law of thermodynamics to find an expression for the Entropy density of the universe in the relativistic limit assuming zero chemical potential :

$$U = TS - pV. \tag{3.12}$$

It is important to note that the volume considered in here we must think of the comoving volume and not the physical volume! Therefore, we have :

$$s = \frac{S}{V} = \frac{\rho + p}{T} \equiv \frac{4}{3}\frac{\rho}{T}, \tag{3.13}$$

where we used the relativistic assumption in the last equality ($p = \frac{\rho}{3}$).

The non-Relativistic Limit :

$$m_i \gg T,\ m - \mu \gg T,\ E = m_i + \frac{p^2}{2m}. \tag{3.14}$$

Remark. *Note that the chemical potential which we ignored in the relativistic limit, can be of importance in the non-relativistic case.*

The number and energy density of different components can be calculated to be :

$$\begin{aligned} N_i &= g_i(\frac{m_iT}{2\pi})^{3/2}exp(\frac{\mu_i - m_i}{T}) \\ \rho_i &= m_iN_i. \end{aligned} \tag{3.15}$$

The expression for pressure is also :

$$p_i \sim g_iN_iT \ll \rho_i. \tag{3.16}$$

Remark. *Note that this is compatible with our equation of state in the previous section. Previously, we said that for non-relativistic matter, we have that $\omega = 0$ in $p = \omega\rho$. Here we also found that $p_i \ll \rho_i$.*

3.2 Cosmic Microwave Background Temperature

The Cosmic Microwave Background (CMB) played (and is playing) a very important role in development of cosmology (you can watch the colloquium by James Peebles for a historical overview). We will study it in detail but now we just want to estimate its temperature. A quick introduction for those who have never even heard of the CMB (which is odd if you have this course :)) would be like this. When the universe started expanding, after some time it cooled enough so that nuclei were formed but because of the still high temperature, there were no neutral atoms. However, there is a point in the history of the universe, where the temperature was low enough for the nuclei and electrons to form neutral atoms. At this point, a flash of light is beamed because of this formation providing us with a picture of the universe from its relatively early days.

Now in order to calculate the temperature of the CMB, we will assume that from its formation till now, the universe has only gone through a radiation dominated phase.

Remark. *Although this assumption is definitely not the most accurate assumption possible, we will see that it still gives a reasonable estimate. This is a common theme in cosmology to make an estimate using crude assumptions and then modifying the result step by step.*

In a radiation dominated universe we have :

$$H = (\frac{8\pi G}{3}\rho_{rad})^{1/2} = \frac{T^2}{M_0} \tag{3.17}$$

with $M_0 = (\frac{45}{4\pi^3 g_* G})^{1/2} = \frac{M_{pl}}{1.66\sqrt{g_*}}$ and g_* is the number of species that dominate in the radiation epoch.

From the previous section we have :

$$a = a_0(\frac{t}{t_0})^{1/2} \Rightarrow H = \frac{\dot{a}}{a} = \frac{1}{2t} \Rightarrow \frac{1}{2t} = \frac{T_{rad}^2}{M_0} \tag{3.18}$$

From our assumption that the universe is radiation dominated, we know that this equation is rather precise when $\rho_m \sim \rho_\gamma$ (and earlier) which corresponds to $t_{eq} \sim 7 \times 10^5 years$ or in terms of temperature $T_{eq} \sim 1eV$. If we put current time $t \sim 10^{10} years$ we get $T_{\gamma,0} = 10K$. Measurements of the CMB temperature show $2.73K$. Our main source of error is because of the matter domination era where $a \sim t^{\frac{2}{3}}$.

Remark. *The reason for which our estimate was good is that :*

$$\begin{aligned} &\frac{t^{\frac{2}{3}}}{t^{\frac{1}{2}}} \sim t^{\frac{1}{6}}, \\ &z_{eq} \sim 10^3. \end{aligned} \tag{3.19}$$

Therefore our estimate is off by a factor of $(10^3)^{1/6}$ *which is not large.*

Remark. *This is in the spirit of early days estimates in the Big Bang theory. Now, we can do much better, but being able to get order of magnitude estimates is very important!*

3.3 g_* in the Standard Model

Our next task is to calculate the g_* parameter using the standard model. There are different sectors which we will take into consideration.

- Scalar
  - Higgs $\sim 1+3$, where the 3 stand for the Goldstone bosons.
- Fermions :
  - Quarks : u, d, c, s, t, b (each has a Left and Right with 3 colours each and their anti-quark counterpart).
  - Leptons : $e_L, e_R, \nu_L^e, \mu_L, \mu_R, \nu_L^\mu, \tau_L, \tau_R, \nu_L^\tau$ plus their anti-leptons.
- Vectors :
  - Photons and 8 gluons (each having 2 polarizations as they are massless)
  - W^+, W^-, Z (each having 3 polarizations as they are massive)

The total would be obtained by summing them all :

$$g_* = 1 + \frac{7}{8}(6\times 2\times 3\times 2) + \frac{7}{8}(9\times 2) + (1+8)\times 2 + 3\times 3 = 106.75, \tag{3.20}$$

at energies more than 172 GeV (more than the most massive particle in the standard model).

Remark. *We should point out that we do not know for a fact that the universe was ever that hot but if it was, then and only then,* $g_* \sim 100$

Remark. *It is important to note that this effective number of species can also be used as a test for theories beyond the standard model.*

3.4 Particle Kinetics

We now study how a gas of particles approaches equilibrium in an expanding universe, that is we first assume a generic distribution function (isotropic and homogeneous) :

$$n(p,t), \tag{3.21}$$

where $\vec{p}$ is the physical momentum. In the absence of interactions, comoving momentum is conserved, therefore :

$$x_p = a x_c \Rightarrow p_p = \frac{p_c}{a}. \tag{3.22}$$

Remark. *The justification for the above equation is as follows. For a massless particle, we remember from last section that the wavelength of a photon decreases as the universe expands. However, for a massive particle we have :*

$$\mathcal{L} = \frac{1}{2} m a^2 \dot{x}^i \dot{x}^i = \frac{p_p^2}{2m}. \tag{3.23}$$

As we know the FRW metric has translational invariance, we have :

$$p_c = a^2(t)\dot{x}^i = \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = a \times p_{ph}. \tag{3.24}$$

Therefore we have :

$$n(p_c, t) = n_0(p_{ph}\frac{a}{a_0}) \tag{3.25}$$

We can now derive the infinitesimal change :

$$\begin{cases} \frac{\partial n}{\partial t} = \frac{\partial n_0}{\partial p_c}.\frac{\partial p_c}{\partial t} = n^{'} p_{ph}\frac{\dot{a}(t)}{a_0}, \\ \\ \frac{\partial n}{\partial p_{ph}} = \frac{\partial n}{\partial p_c}.\frac{\partial p_c}{\partial p_{ph}} = n^{'}\frac{a(t)}{a_0} \Rightarrow n^{'} = \frac{a_0}{a(t)}\frac{\partial n}{\partial p_{ph}} \end{cases} \tag{3.26}$$

$$\Rightarrow \frac{\partial n}{\partial t} = \frac{a_0}{a(t)}.\frac{\partial n}{\partial p_{ph}}p_{ph}\frac{\dot{a}(t)}{a_0} = Hp_{ph}\frac{\partial n}{\partial p_{ph}}, \tag{3.27}$$

$$\Rightarrow \frac{\partial n}{\partial t} - Hp_{ph}\frac{\partial n}{\partial p_{ph}} = 0. \tag{3.28}$$

If we integrate over p we get ($N = \int nd^3p$):

$$\begin{aligned} &\frac{dN}{dt} - \int \frac{d^3p}{(2\pi)^3}Hp_{ph}\frac{\partial n}{\partial p_{ph}} = 0 \\ \Rightarrow &\frac{dN}{dt} + 3HN = 0, \end{aligned} \tag{3.29}$$

where we used integration by parts in the last step.

Remark. *It is important to bear in mind that this equation was derived for the case of non-interacting particles.*

3.4.1 Adding Interactions

In our next step,we would like to include the interactions among particles. We will focus on $2 \Rightarrow 2$ scattering (collisions) that can transfer momentum and hence change the phase space distribution.

In order to include interactions, we use collision integrals.

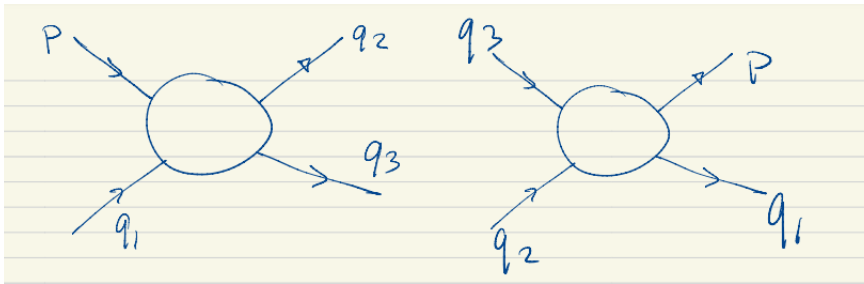
$$\frac{\partial n}{\partial t} - Hp\frac{\partial n}{\partial p} = I_{col.}. \tag{3.30}$$

The above equation is called the Boltzmann equation.

In general, the collision integral is of the following form (this comes from the BBGKY hierarchy):

$$\begin{aligned} I_{col.} = -\frac{1}{2p^0}\int \frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}\frac{d^3q_3}{(2\pi)^3}(2\pi)^4\delta^{(4)}(p + q_1 - q_2 - q_3)|M_{fi}|^2\times \\ (n(p)n(q_1)[1 \pm n(q_2)][1 \pm n(q_3)] - n(q_3)n(q_2)[1 \pm n(p)][1 \pm n(q_1)]), \end{aligned} \tag{3.31}$$

where $|M_{fi}|^2$ is the scattering amplitude. The first term corresponds to the right process and the second term corresponds to the left process :



Remark. *It is in general hard to solve the Boltzmann equation with the following collision integral as it involves derivatives and integrals and is non-linear. That is why we use approximations.*

Remark. *In our collision integral it is obvious that we have not considered the possibility of pair production. That is to say that particle number is conserved.*

Remark. *Note that the Fermi-Dirac statistics as well as the Bose-Einstein statistics are solutions of the collision integral, meaning $I_{col.}(n_{F/B}) = 0$.*

The final remark allows us to use the relaxation time approximation. We use $n \sim n_{eq}$. We can now Taylor expand the collision integral in terms of this parameter :

$$I_{col.}(p) = (n - n_{eq})(-\Gamma) + ... \tag{3.32}$$

where Γ can be interpreted as the rate of reaction:

$$\Gamma = \tau^{-1} = (< \frac{\lambda}{v} >)^{-1}, \tag{3.33}$$

where τ is the mean free time, λ is the mean free path, v is the relative velocity and the average is taken using the equilibrium distribution (taking the average with the original distribution would give a subleading term). We can also take the cross section as σ and have :

$$\lambda = (\sigma N)^{-1}, \tag{3.34}$$

with N being the number of particles in the volume. Therefore we can say :

$$\Gamma = \tau^{-1} = (< \frac{\lambda}{v} >)^{-1} \approx < \sigma N v > . \tag{3.35}$$

Now if we again take the integral of the Boltzmann equation we have :

$$\frac{dN}{dt} + 3HN = -\Gamma(N - N_{eq}). \tag{3.36}$$

3.4.2 Freeze in Freeze out

Our Boltzmann equation has two terms : one which drives the theory away from thermal equilibrium and one that drives towards it. However, note that they both depend on temperature.

- σ is a function of relative and average speed which both depend on temperature.

- The Hubble parameter is also a function of time and therefore a function of temperature (remember that we treat time and temperature on the same footing during the evolution of the universe).

There are two asymptotic regimes to be considered :

$$N = \begin{cases} \approx N_{eq} & \Gamma \gg H \\ \sim N_{eq}(T_*)(\frac{a(t_*)}{a(t)})^3 & H \gg \Gamma \end{cases} \tag{3.37}$$

where T_* is the temperature at which two effects are equal.

T_* is also called the **Freeze out** or **Freeze in** temperature depending on whether $\frac{\Gamma}{H}$ is approaching 1 from below or above.

Remark. *Note that at very late universe things always freeze out.*

Let us now apply all the knowledge we gained in an example.

The electron-positron gas : We will consider the thermalization of electrons, positrons and photons (γ) in the radiation domination era ($a \sim t^{1/2}$). The cross section of $e^+e^- \Leftrightarrow \gamma\gamma$ can be computed using quantum electrodynamics which we take for granted :

$$\sigma = \begin{cases} \frac{1}{2v}\pi r_e^2 & v \ll 1 \\ \\ \frac{m_e^2}{E^2}\pi r_e^2 \log(\frac{4E^2}{m_e^2} - 1) & v \approx 1 \end{cases} \tag{3.38}$$

where $r_e = \frac{\alpha}{m_e}$, $\alpha = \frac{e^2}{4\pi}$ and by E we mean the center of mass energy.

Now consider the different regimes :

- $(T \lesssim m_e \Leftrightarrow v \ll 1) \Rightarrow N_i = g_i(mT)^{3/2}e^{-\frac{m}{T}}$.
- $(T \gg m_e \Leftrightarrow v \approx 1) \Rightarrow N_i = \frac{3}{4}\frac{\xi(3)}{\pi^2}g_iT^3$.

plugging in Γ :

$$\Gamma = <\sigma N v> = \begin{cases} \sim r_e^2(m_eT)^{3/2}e^{-\frac{m_e}{T}} & T \lesssim m_e \\ \sim \alpha^2 T & T > m_e \end{cases} \tag{3.39}$$

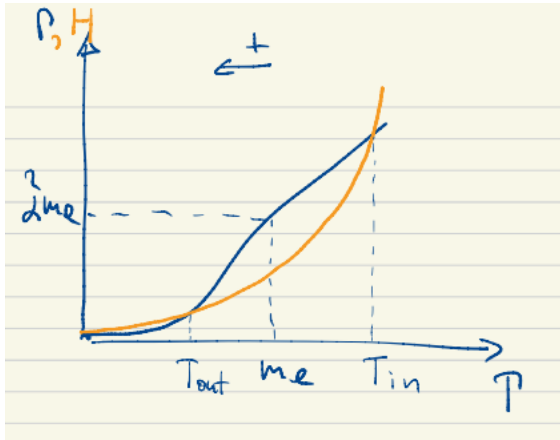
where we ignored the log in the cross section as well as all order one constants.

Remark. *Note that these limits agree when $T \sim m_e$.*

We know that the Hubble parameter's relation to temperature is as :

$$H(T) = \frac{T^2}{M_0}. \tag{3.40}$$

Now we can draw a diagram of both the Hubble parameter and Γ :



We can find the freeze-in and freeze-out temperatures as follow :

$$T_{in}: \quad \alpha^2 T = \frac{T^2}{M_0} \Rightarrow T_{in} = \alpha^2 M_0 \sim 10^{14} GeV. \tag{3.41}$$

For T_{out}, it needs a little more work :

$$T_{out}: \quad r_e^2 (m_e T)^{3/2} e^{-\frac{m_e}{T}} = \frac{T^2}{M_0}. \tag{3.42}$$

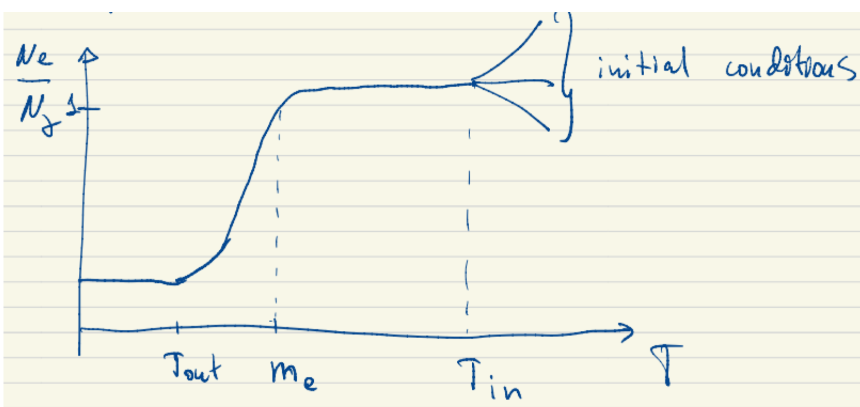
Now define : $x = \frac{m_e}{T_{out}}$. Then we have :

$$e^{-x} x^{1/2} = \frac{m_e}{\alpha^2 M_0} \approx 10^{-18}, \tag{3.43}$$

where numerically corresponds to $x = 40$. Therefore :

$$T_{out} \approx \frac{m_e}{40} \approx 10 KeV. \tag{3.44}$$

We can assume that photons are in thermal equilibrium, their number density scales as $N_\gamma \sim a^{-3}$. This is consistent with the fact that their distribution stays thermal and only their temperature varies.



Remark. *The paramount aspect of all the above calculations is that the final density of electrons and positrons is **independent** of the initial conditions!*

3.5 Recombination

In the early universe, atoms are highly ionized and there are a lot of free charges. This means that this plasma is not transparent to photons. What we mean by this, is that before the photon reaches the observer, it collides with another ion. So when does the universe become transparent to photons? This happens only slightly after the $e\gamma \to e\gamma$ freezes out.

In general, at later epochs, the universe becomes transparent and the photons travel freely (we can see stars from galaxies far far away!). The transition between the two epochs is called recombination. Maybe it is not the most suited name since the ions are not **re**combining, they are being combined for the first time. Be it as it may, it is called recombination :). This is because at some stage, the temperature falls below the binding energy of the Hydrogen atom, so it becomes favourable for them to form :

$$p + e \to H + \gamma, \tag{3.45}$$

where the binding energy is around $I = 13.6eV \sim 1.5 \times 10^5 K$.

Remark. *Note that at around this time, all anti-matter has already annihilated (previously we calculated that the electron-positron freeze-out happens at* $10keV$*).*

Furthermore, after the ions form neutral atoms, photons propagate freely and the universe becomes transparent. The time-slice when the "last" scattering occurs is called the **Last Scattering Surface**. The CMB is a snapshot of the last scattering surface!

In more details, the number densities of each component follow Saha's equation (assuming thermal equilibrium) :

$$\frac{N_e N_p}{N_H} = (\frac{m_e T}{2\pi})^{3/2} e^{-\frac{I}{T}}. \tag{3.46}$$

In order to derive Saha's equation, we emphasize again that we assume thermal equilibrium. This is of course not the most precise way of modelling, but it turns out that the big picture characteristics that we derive will remain intact.

Previously we derived the number density of fermions, therefore we have :

$$\begin{aligned} N_e &= g_e(\frac{m_e T}{2\pi})^{3/2} e^{-\frac{m_e-\mu_e}{T}}, \\ N_p &= g_p(\frac{m_e T}{2\pi})^{3/2} e^{-\frac{m_p-\mu_p}{T}}, \\ N_H &= g_H(\frac{m_e T}{2\pi})^{3/2} e^{-\frac{m_H-\mu_H}{T}}, \end{aligned} \tag{3.47}$$

where $g_e = g_p = 2$ and $g_H = 4$. Moreover, chemical equilibrium means : $\mu_p + \mu_e = \mu_H$ ($\mu_\gamma = 0$). Otherwise the reaction would change the distribution. Now we can compute $\frac{N_e N_p}{N_H}$.

$$\Rightarrow \frac{N_e N_p}{N_H} \approx (\frac{m_e T}{2\pi})^{3/2} e^{-\frac{(m_e+m_p-m_H)}{T}}, \tag{3.48}$$

which is exactly Saha's equation.

Our upcoming goal is to compute $\Gamma =< \sigma_\gamma N_e v >$. Moving forward, we follow the following logic : We assume that the plasma is in equilibrium and look for the temperature of decoupling of Thompson scattering $e\gamma \to e\gamma$. Note that this is a simplification. (For a more in depth treatment, see Rubakov, Gorbunov vol I chapter 6) Nevertheless, let us proceed. Neutrality of the universe implies $n_p = n_e$. Substituting into Saha's equation, we get :

$$N_e = \sqrt{N_H}(\frac{m_e T}{2\pi})^{3/4} e^{-\frac{I}{2T}}. \tag{3.49}$$

Now we have to determine the number of baryons $N_p + N_H = const$. As we will see soon, $N_p \ll N_H$ at the temperature of interest meaning that most atoms recombined.

Additionally, the ratio of Baryons to photons is a very important factor in cosmology. It is given by :

$$\eta = \frac{N_B}{N_\gamma} = 5 \times 10^{-10}. \tag{3.50}$$

Note that η is determined from various different data such as CMB, BBN, etc., we will discuss it later. Moreover, the ratio of Helium to Hydrogen is 25% to 75% (determined by BBN).

Now as for the cross-section, we have :

$$\sigma_{\gamma e} = \frac{8\pi\alpha^2}{3m_e^2}, \quad \sigma_{\gamma p} = \frac{8\pi\alpha^2}{3m_p^2}. \tag{3.51}$$

This is why only the electron Thompson scattering is relevant since the proton scattering is suppressed by factors of proton mass. Now we will use $H = \frac{T^2}{M_0}$. Even though we will end up in matter domination era and this formula was obtained for radiation domination era, this is a good approximation.

$$\Gamma = H \Rightarrow B\frac{T_*^{9/4}}{m_e^{5/4}} e^{-\frac{I}{2T_*}} = \frac{T_*^2}{M_0}, \tag{3.52}$$

where $B = \frac{8\pi\alpha^2}{3}\sqrt{\eta}(\frac{2\xi(3)}{\pi^2})^{1/2}(\frac{1}{2\pi})^{3/4}$. To solve this equation, define $x = \frac{I}{2T}$.

$$\Rightarrow x^{-1/4}e^{-x} = 1.2 \times 10^{-12} \Rightarrow x \approx 27 \Rightarrow T \approx 0.25eV \sim 3000k. \tag{3.53}$$

We know that the CMB temperature today is $2.73k$. Therefore we have :

$$z_{dec.} = \frac{T_*}{T_{CMB}} \approx 1100. \tag{3.54}$$

Remark. *Compare with the above result that matter-radiation equation happens at $z = 3000$ and convince yourself that our previous approximations make sense.*

3.6 Neutrinos

Neutrinos (ν) are very weakly interacting (at low energies) $E \sim 1MeV$:

$$\frac{\sigma_\nu}{\sigma_e} \approx 10^{-18}. \tag{3.55}$$

However, at early times, neutrinos were thermalized just like photons and then they decoupled. Following the same logic, there is also a surface of last scattering for neutrinos.

Remark. *It is noteworthy that at some point, scientists guessed neutrinos to be candidates for dark matter, but now we know it is not the case due to more precise measurements.*

Let us calculate the temperature and number density of neutrinos. Firstly, this is the neutrino cross-section :

$$< \sigma_{e\nu} v >\sim G_F^2 E^2, \quad G_F \approx 10^{-5} GeV^{-2}. \tag{3.56}$$

Neutrinos are relativistic at decoupling but non-relativistic now (one of them might be massless). Nevertheless, they are described by relativistic distribution even at late times.

$$\Rightarrow N_\nu \sim T^3 \Rightarrow \Gamma =< \sigma_{\nu e} v N >\sim G_F^2 T^2 T^3, \quad H = \frac{T^2}{M_0}, \tag{3.57}$$

therefore we have $T_*^\nu = (G_F^2 M_0)^{-1/3} \sim 2MeV$.

The effective temperature of neutrino distribution for now is :

$$T_{\nu,0} = T_*^\nu \frac{a(T_*^\nu)}{a(T_0)}. \tag{3.58}$$

Remark. *This last equation is the reason why neutrinos have been ruled out as candidates of dark matter.*

Naively, one might say : $T_{\nu,0} = T_{\gamma,0} = T_{CMB}^0$. However, upon inspection we see that this is not quite precise! The reason is because $e^+e^- \to \gamma\gamma$ happens after T_*^ν, so $e^+e^- \to \gamma\gamma$ heats up photons! To compute how much this difference is, we can use entropy conservation :

$$g_*^{e\gamma} a^3 T^3 = const, \tag{3.59}$$

where :

$$g_*^{e\gamma} = \begin{cases} g_\gamma + g_{e^-,e^+} = 2 + \frac{7}{8}(2+2) = \frac{11}{2} & T \gg m_e \\ 2 & T = T_0 \end{cases} \tag{3.60}$$

Therefore we have :

$$\begin{aligned} &\frac{T_{\gamma,0}}{T_{\nu,0}} = [\frac{g_*^{e\gamma}(T_\nu^*)}{g_*^{e\gamma}(T_0)}]^{1/3} = (\frac{11}{4})^{1/3} \approx 1.4 \\ \Rightarrow &T_{\nu,0} = 2k. \end{aligned} \tag{3.61}$$

As a last word, we elaborate on the cosmological bound on Neutrino masses. Here, we assume that neutrinos are non-relativistic. Note that this is just an assumption!

$$\begin{aligned} \rho_{\nu,0} &= m_\nu N_{\nu,0}, \\ \frac{N_\nu}{N_\gamma} &= \frac{3}{4}\frac{g_\nu}{g_\gamma}(\frac{T_\nu}{T_\gamma})^3 \approx \frac{3}{22}. \end{aligned} \tag{3.62}$$

Note that the second equation is true for every neutrino species.

$$\Rightarrow \rho_{\nu,0}^{tot} = \sum_i m_{\nu_i} \frac{3}{22} N_{\gamma,0} \lesssim \rho_{DM,0}, \tag{3.63}$$

where $\rho_{DM,0}$ is the dark matter abundance. Note that this is a very naive inequality, meaning that we could do better in principle, but as a first approximation this is good enough. Furthermore, we know that $\rho_\gamma \sim T^4$ and $\rho_{DM} \sim z_{eq}\rho_\gamma$ with $z_{eq} \approx 3000$. Substituting all in, we have :

$$\sum m_{\nu_i} \sim 10eV. \tag{3.64}$$

A more detailed consideration would give :

$$\sum m_{\nu_i} \sim 0.2eV. \tag{3.65}$$

Remark. *It is important to note that historically, even the 10eV bound on the neutrino mass was significantly better than the bound found using particle colliders!*

4 Big Bang Nucleosynthesis

Big bang nucleosynthesis (BBN) is the process of formation of light nuclei ($H, He, D, Li, ...$) from protons and neutrons of the primordial plasma. It occurred at $T \sim 1 - 0.1 MeV$. Note that atoms form much later during and after recombination and all heavier elements are formed even later in stars. As a result of BBN we get 75% Hydrogen, 25% Helium, and $\sim 2 \times 10^{-5}$ of Deutirium. These abundances are measured and as they depend significantly on various parameters both from particle physics and cosmology such as $\eta_B, g_*, ...$, these parameters get constrained.

Remark. *These are the most primordial direct evidence we have from the universe. Basically, before this, anything could have happened, but afterwards, it must match these abundances! This is actually a very powerful way of testing different "beyond the standard model" theories!*

4.1 Neutron-Proton ratio, Freeze-out of Neutrons

Remember that $m_p \sim m_n = 1GeV, m_n - m_p = 1.3MeV$. We can check when the reaction

$$p + e \to n + \nu_e, \tag{4.1}$$

will go out of equilibrium. Since this reaction is possible through weak interactions, T^* will be close to that of neutrinos :

$$T_\nu^* \sim (G_F^2 M_0)^{-1/3} \sim 2MeV. \tag{4.2}$$

A more concise calculation gives $T_n^* \sim 0.8MeV$.

Remark. *This result is an incredible coincidence! The fact that T_n^* and $m_n - m_p$ are comparable, is a complete coincidence! Note that one comes from strong interactions and particle physics considerations, while the other one involves the Hubble parameter and the Planck mass! Note that if this was not the case, all neutrons would decay and the universe would only consist of Hydrogen. A very boring universe indeed!*

Let us compute the abundances, assuming thermal equilibrium (neutrino's just froze out but they are still in thermal equilibrium). We have :

$$N_n \sim e^{-\frac{m_n - \mu_n}{T}}, \quad N_p \sim e^{-\frac{m_p - \mu_p}{T}}. \tag{4.3}$$

Moreover, as we assumed thermal equilibrium, we have :

$$\mu_p + \mu_e = \mu_n + \mu_\nu. \tag{4.4}$$

This gives :

$$\frac{N_n}{N_p} = e^{-\frac{I}{T} + \frac{\mu_e - \mu_\nu}{T}}. \tag{4.5}$$

Furthermore, we assume that μ_ν is small. We could have not used this assumption and kept the chemical potential for neutrinos until the end of our calculations and then fine tune it with observation, but since we know a priori that it is in agreement with observation, we use this assumption as it makes calculations easier.

For the electron chemical potential, we have :

$$\frac{\mu_e}{T} \approx \frac{N_{e^-} - N_{e^+}}{N_{e^-} + N_{e^+}} \approx \eta_B \sim 10^{-10}. \tag{4.6}$$

The reason for the first equality comes from the meaning of chemical potential as the tendency for the reaction to move towards one side of the reaction. The second equality is a consequence of neutrality of the universe, and the fact that at this epoch, electrons and positrons are both relativistic and therefore have the same distribution as photons. We finally have :

$$\frac{N_n}{N_p} \approx e^{-\frac{m_n - m_p}{T}} \approx e^{-\frac{1.3}{0.8}} \approx \frac{1}{5}. \tag{4.7}$$

Remark. *Note that the timescale at which we are doing our computations (age of the universe), is around 1 second. This is why neutron decay which has a time scale of 15 minutes can be ignored.*

4.2 Nuclear reactions

The next step is to wait until nuclear reactions become important :

$$\begin{aligned} p + n &\longleftrightarrow D + \gamma, \\ D + D &\longleftrightarrow {}^3H + p; \quad D + D \longleftrightarrow {}^3He + n, \\ {}^3H + D &\longleftrightarrow {}^4He + n, \\ {}^3He + D &\longleftrightarrow {}^4He + p. \end{aligned} \tag{4.8}$$

By definition, the binding energy of an atom is :

$$I = Zm_p + (A - Z)m_n - M. \tag{4.9}$$

As we will see, the important parameter would be $\frac{I_A}{A}$:

$$\frac{I_A}{A} = \begin{cases} D & 1.1MeV \\ {}^3H & 2.3MeV \\ {}^4He & 7MeV \\ {}^12C & 7.7MeV \\ {}^26Fe & 8.8MeV \end{cases} \tag{4.10}$$

It is important to note that Fe is the most stable atom in the periodic table.

The reactions active during BBN never produce ${}^{12}C$, but they produce ${}^4He$ in which most of the neutrons end up being. Before that however, Deuterium has to be produced. Let us estimate when the following reaction gets shifted to the right (which means that it is favourable to have Deuterium instead of just having protons and neutrons) :

$$p + n \longleftrightarrow D + \gamma. \tag{4.11}$$

For that, let us compute temperature at which abundance of Neutrons and Deuterium are equal. This is very similar to what we already did for the CMB. The only conceptual difference is that the CMB was done on the atomic level whereas here we are computing parameters on the nuclear level.

$$\frac{N_p N_n}{N_D} = (\frac{m_p T}{2\pi})^{3/2} e^{-\frac{I_D}{T}}. \tag{4.12}$$

Furthermore, we use the usual logic that $N_p \sim \eta_B N_\gamma \sim 0.24T^3\eta_B$. The point where we have the reaction rate is equal to the Hubble rate, just like in the CMB, is called the T_{NS} :

$$\Gamma(T_{NS}) = H(T_{NS}) \Rightarrow T_{NS} \sim 70KeV. \tag{4.13}$$

Now we face something different than the CMB. The CMB and the last scattering surface do not happen at the same time. The CMB happens and after some time we have the last scattering surface. The reason why we did not emphasize on this point in the past is that there is really not very interesting physics happen in between the two. However, here there are interesting things happening. This is because of the short lifetime of Neutrons! Therefore from the 0.8 MeV to the 70 KeV temperature, which is around $T_{NS} \sim 4$ minutes, we should account for decaying Neutrons. Therefore, at $T_{NS} \sim 70KeV$, the Neutron-Proton ratio is :

$$\frac{N_n}{N_p} \sim e^{-\frac{I}{T_n^*}} \times e^{-\frac{T_{NS}}{\tau_n}} \approx \frac{1}{7}, \tag{4.14}$$

where τ_n is the Neutron decay time-scale. This means that from the time that Neutrons froze out until the time which Deuterium production becomes relevant, we lost some of the Neutrons due to decay. Now, we are ready to start calculating the Helium abundance assuming equilibrium.

Remark. *Note that* $e^{-\frac{I}{T_n^*}} = \frac{1}{5}$. *Therefore by what we did, it is implied that BBN is actually sensitive to the difference of 5 and 7!*

Remark. *At this point, you might be confused as we are ignoring some processes while cherry-picking some other. This is only due to some combination of importance plus easiness of computation.*

4.2.1 ^4He Abundance

Define $x_A = \frac{AN_A}{N_B}$ where N_B is the number density of Baryons. Therefore, by definition we have :

$$\sum_A x_A = 1. \tag{4.15}$$

Now, as we saw that the nuclear binding energy per nucleon is largest for ^4He (amongst the considered reactions of Hydrogen, Deuterium, Tritium and Helium) and considering the fact that the number of protons is higher than the number Neutrons by a factor of 7, all Neutrons end up in Helium. Therefore, we have :

$$N_{^4He} = \frac{1}{2} N_n. \tag{4.16}$$

This approximation is justified as we already see in experiment that the next highest abundance is Deuterium which is suppressed by a factor of 10^{-5}.

$$\Rightarrow x_{^4He} = \frac{4N_{^4He}}{N_p + N_n} = \frac{2.\frac{1}{7}}{1+\frac{1}{7}} \approx 25\%. \tag{4.17}$$

Now, let us do the same calculation a bit more carefully. Since we are assuming that all reactions are at equilibrium :

$$\mu_A = Z\mu_p + (A - Z)\mu_n. \tag{4.18}$$

Our assumption of equilibrium here means that we are basically considering only one reaction that Z protons and $(A - Z)$ neutrons are all combined in one reaction to form the desired nuclei. Therefore, a modification of Saha's equation will be :

$$N_A = (N_p)^Z (N_n)^{A-Z} (\frac{1}{m_p T})^{\frac{3}{2}(A-1)} e^{\frac{I_A}{T}}. \tag{4.19}$$

Moreover, we write :

$$N_B \approx \eta_B T^3 \approx N_p, \tag{4.20}$$

where we disregarded the $\frac{1}{7}$ ratio of neutrons. Again using the definition of x, we have :

$$x_n = x_{^4He}^{\frac{1}{2}} \eta_B^{-3/2} (\frac{T}{m_p})^{-9/4} e^{-\frac{I_{^4He}}{T}}. \tag{4.21}$$

Again, remember that as we know that Helium has the highest abundance, we expressed only in terms of Helium.

Now, had we used the same logic for a general element instead of neutrons, we would have gotten :

$$x_A = [\eta_B (\frac{T}{m_p})^{\frac{3}{2}}]^{\frac{3}{2}z - \frac{1}{2}A - 1} exp(\frac{I_A - I_{^4He}(\frac{A-Z}{2})}{T}). \tag{4.22}$$

Now the sign of the exponent is determined by the following expression :

$$\frac{I_A}{A - Z} - \frac{I_{^4He}}{4 - 2}, \tag{4.23}$$

which is why we mentioned before that it is the binding energy per nucleon that matters as opposed to the binding energy itself. Here we found out that it is actually the binding energy per neutron which matters. It is actually a coincidence that both for the binding energy per nucleon and per neutron, ^{4}He has the maximum value between the light elements. Here is the values for the binding energy per neutron (compare with the binding energy per nucleon which was given before) :

$$\frac{I_A}{A - Z} = \begin{cases} D & 2.3MeV \\ ^3H & 4.2MeV \\ ^3He & 7.7MeV \\ ^4He & 14.2MeV \\ ^6Li & 10.6MeV \\ ^{12}C & 15.4MeV \end{cases} \tag{4.24}$$

Moreover, note that we are now working at temperatures of around $70KeV$, whereas the binding energies are of $1MeV$, therefore, their abundances are very small! For example, if we plug in Deuterium into this formula, we get $x_D \sim e^{-120}$. This result is of course absurd because if this was the case, we would never measure any Deuterium in the universe! The actual result is around e^{-10} which is measurable. We will now compute this abundance by considering departures from thermal equilibrium.

4.2.2 Abundance of Deuterium

Deuterium production process :

$$p + n \leftrightarrow D + \gamma. \tag{4.25}$$

Deuterium destruction (burning) :

$$\begin{aligned} D + D &\leftrightarrow {}^3\text{He} + n \\ D + D &\leftrightarrow T + p \end{aligned} \tag{4.26}$$

The point of this out of equilibrium calculation is that Deuterium freezes out before it all burns. Following the same logic of before :

$$\frac{dN_D}{dt} + 3HN_D = - < \sigma v N_D > N_D. \tag{4.27}$$

It is noteworthy that we are working in the regime were :

$$N_D^{eq} \ll N_D \ll N_{^4\text{He}}. \tag{4.28}$$

Now we have :

$$\sigma v = 3 \times 10^{-17} \frac{cm^3}{s}, \quad T \sim T_{NS}. \tag{4.29}$$

As usual we consider the case where $H \approx \Gamma$.

$$H = \frac{T^2}{M_0} \approx \sigma v N_D, \quad N_p = \eta_B T^3. \tag{4.30}$$

$$\Rightarrow \frac{N_D}{N_p} = \frac{1}{\sigma v \eta_B M_0 T_{NS}} \approx 10^{-5}. \tag{4.31}$$

Remark. *We have arrived at a very interesting observation. The fact that Deuterium abundance is inversely proportional to the baryon to photon ratio is actually one of the ways that we can fix η_B observationally.*

5 Baryogenesis

Baryogenesis is the term pointing the process responsible for the obvious difference in the number of baryons and anti-baryons. There is no known process (such as BBN or CMB) which is responsible for this. However, there are several conjectures about how it happened.

Most often, people assume an initial state of symmetry under baryon $\leftrightarrow$ anti-baryon and it is the evolution which produces the asymmetry $n_B > n_{\bar{B}}$. We should bear in mind that this is in fact also an assumption. What happens is that most of the B and $\bar{B}$ annihilate leaving :

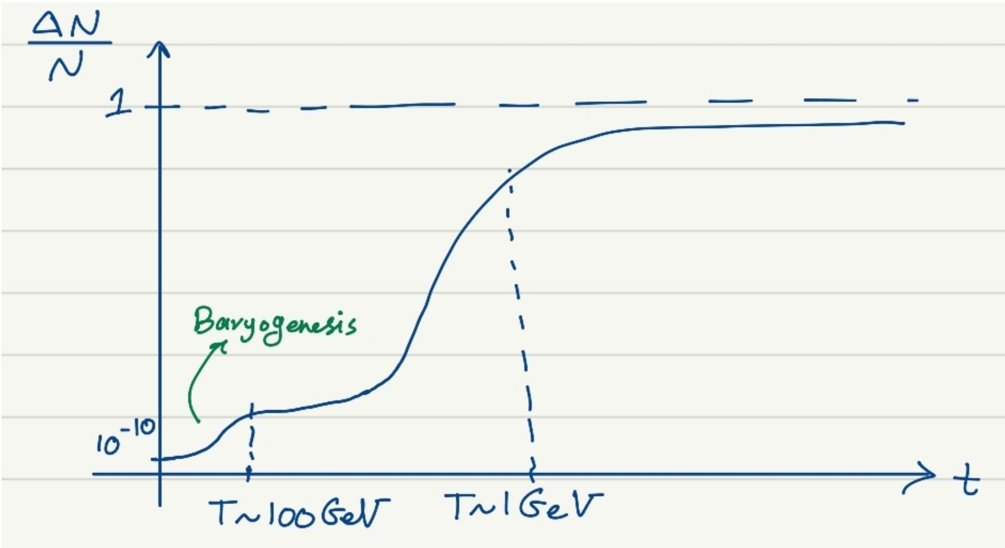
$$n_B^{final} = \Delta n. \tag{5.1}$$

Note that since below $T \sim 100GeV$ baryon symmetry is a good symmetry (meaning that it is conserved), we do not need to worry about n_B changing after that.

Remark. *Note that the reason behind $\eta_B \ll 1$ is actually Baryogenesis as well. Almost all baryons and anti-baryons collide and turn into photons which makes the baryon to photon ratio very small. In equations we mean :*

$$[T \gg 1GeV : \quad N_B \sim N_{\bar{B}} \sim N_\gamma \sim T^3] \Rightarrow \frac{\Delta n}{n_B} \sim \eta_B \sim 10^{-10}. \tag{5.2}$$

Figuratively :



Sakharov's Conditions : For baryogenesis it is necessary to have :

- Baryon number violation
- Departure from thermal equilibrium
- Violation of C and P.

By C, P and T we mean :

- $P : \vec{x} \to -\vec{x}$

- $T : t \to -t$
- $C : B \to -\bar{B}$

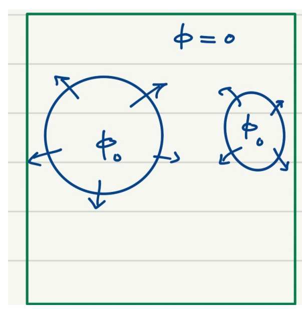
CPT is an exact symmetry which implies that masses of particles and anti-particles (as well as their interactions) are the same. Notably, each of these individual symmetries are broken. However, breaking of CP (or equivalently T) is weaker than that of just C or P. (This observation was first made through Kaon decay)

Interestingly, the Standard model of particle physics has all of the ingredients in Sakharov's condition! The problem lies in the fact that the effect is too small. This is because Baryon number violation is a non-perturbative effect (just like quantum tunneling in ordinary quantum mechanics called Sphaleron). These non-perturbative effects happen such that :

$$\Gamma \sim e^{-\frac{E_{sph}}{T}}, \quad E_{sph} \sim 100GeV. \tag{5.3}$$

This process is in thermal equilibrium until $T \sim 100GeV$. Moreover, B-L is an exact symmetry in the standard model. Therefore, if B asymmetry is generated at times earlier than 100 GeV, one needs to create a B-L asymmetry, otherwise sphalerons will erase just B asymmetry. Usually this is done by adding heavy particles that decay in ways which B-L is not kept invariant. Another way is to generate asymmetry just with particles from the standard model. (This idea although very elegant does not seem to work)

ElectroWeak Baryogenesis : The main problem is to satisfy the 2nd Sakharov condition. If the Electroweak phase transition was of first order, there would be enough deviation from thermal equilibrium :



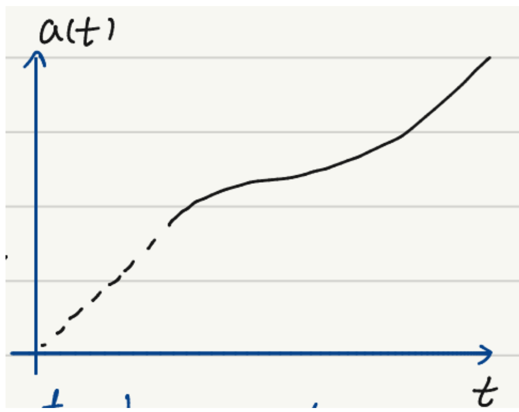
The Sphaleron process (mentioned above) could generate B asymmetry (creating some L asymmetry). The reason why this does not work is because the ElectroWeak phase transition turns out to be of second order.

6 Dark Matter

In this section, we will explain the reasons why we believe dark matter exists and the candidates we have for them.

6.1 DM Existence

How do we know there is dark matter? FRW evolution of the universe tells us the total amount of matter, and we can also measure the amount of Baryons because they interact through the standard model (basically it is because they are visible).



Historically, it happened through a more direct observation. The rotation curves of stars in galaxies (this can be done because most baryons are in the stars and we can see them). Baryon distribution is observed to be :

$$I_B = I_0 exp(\frac{r}{r_0}) \tag{6.1}$$

Using basic Newtonian dynamics and gravity we have :

$$\begin{aligned} mv^2 &\sim G\frac{Mm}{r} \\ \Rightarrow v &\sim \frac{1}{\sqrt{r}}. \end{aligned} \tag{6.2}$$

Remark. *Note that this Newtonian limit should be considered for the case of $r \gg r_0$.*

However, the velocity profile is observed to be :

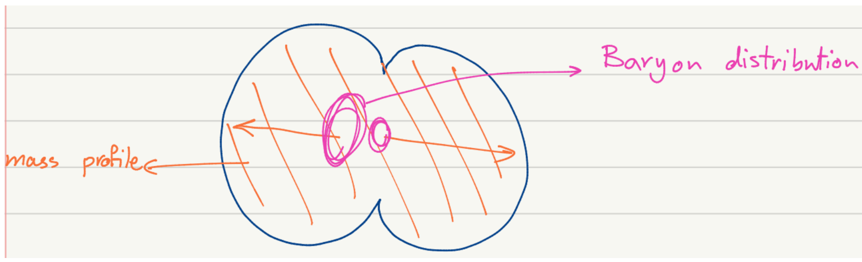
$$v \sim const. \tag{6.3}$$

This means that in case we do not consider any modified theory of gravitational dynamics, we must have $M \sim r$ which suggests $\rho_{dark} \sim \frac{1}{r^2}$. Furthermore, from observation we have :

$$\rho_{dark}(r) = \frac{r_c^2}{r^2 + r_c^2}. \tag{6.4}$$

Additionally, gravitational lensing allows to measure the total mass of distant galaxy clusters which is consistent with other dark matter proportion relative to other tests.

Remark. *One of the most important constraints on dark matter comes from "Bullet clusters". It heavily restricts the self interaction of dark matter.*



All in all, our entire data from BBN, CMB and the Large Scale Structure (LSS) point to the same approximate amount of dark matter. Therefore, we state that we have observational evidence from very different physical effects and cosmological epochs! This is the reason behind the fact that we do not think the answer to the dark matter conundrum comes from modified gravity.

6.2 Dark Matter Candidates

6.2.1 Neutrino Dark Matter

Dark matter should be stable and very weakly interacting. By weakly interacting we mean with ordinary matter and also with itself, from our observations on bullett clusters. Our first guess would come from the only particle of the standard model satisfying these two conditions which are Neutrinos.

In order to further investigate this candidate, note that Neutrinos are fermions. Therefore by Pauli exclusion principle, we know that the total number of neutrinos are less than the available phase space :

$$N_\nu \leq \frac{1}{(2\pi)^3} \int d^3p d^3x n \sim p^3 r^3. \tag{6.5}$$

Furthermore, we can substitute $p = mv$, $M_\nu \lesssim m_\nu N_\nu \sim m_\nu^4 v^3 r^3$, where M_ν is the total dark matter mass if neutrinos are dark matter and m_ν is the neutrino mass.

$$\Rightarrow v^2 \lesssim \frac{GM_\nu}{r} \Rightarrow m_\nu \gtrsim (\frac{1}{Gvr^2})^{1/4} = 120eV(\frac{100\frac{km}{s}}{v})^{1/4}(\frac{1kpc}{r})^{1/2} \tag{6.6}$$

For our own galaxy we have $v \sim 220\frac{km}{s}$ and $r_c \sim 10kpc$. Therefore, we have $m_\nu \gtrsim 30eV$ which is inconsistent with the neutrino mass we found in earlier chapters (0.2 eV). If we calculate the same mass constraint for dwarf galaxies we find $m_\nu \gtrsim 300eV$!

Remark. *Our calculations mean that any fermionic dark matter must satisfy the same mass bound.*

Similar considerations for bosonic dark matter gives a much less restrictive bound as our approximation comes from the fact that the Compton wavelength cannot be larger that the halo radius :

$$\lambda_c = \frac{\hbar}{mc} \lesssim r_c \Rightarrow m_b \gtrsim 10^{-20}eV. \tag{6.7}$$

6.2.2 Axion Dark Matter

The next candidates are QCD Axions. This candidate is interesting as it is not only used to explain dark matter but also strangeness of other phenomena.

In the standard model of particle physics, there is a strange occurrence of very little CP violation in QCD and relatively large CP violation in the electroweak sector. The addition of Axions somehow solves this problem as Axions violate CP themselves. Due to various experiments (including the Axion Dark Matter eXperiment (ADMX)), Axions can have a mass in the range :

$$10^{-11}eV \leq m_A \leq 10^{-2}eV. \tag{6.8}$$

Axions are also very weakly interacting which makes them a suitable candidate for dark matter while posing challenges to experimentally measuring them. Axions can also be dark matter if $m_A \lesssim 10^{-5}$. We will further explain where these results come from. The potential for the Axion is :

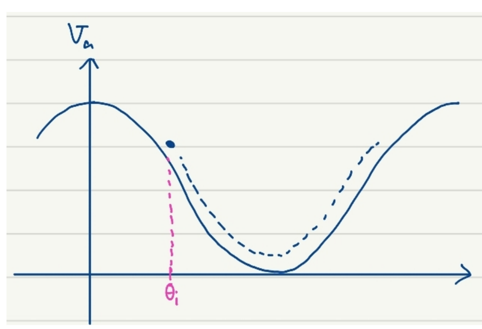
$$V_A|_{T=0} = -\Lambda^4_{QCD}(\cos(\frac{A(\vec{x},t)}{f_A}) + 1) \sim m_A^2 A^2, \tag{6.9}$$

where Λ_{QCD} is the QCD mass scale, $A(\vec{x},t)$ is the Axion field, f_A is some constant and $m_A = \frac{\Lambda^4_{QCD}}{f_A^2}$. We also believe that $f_A \lesssim M_{Pl}$ and therefore we have $m_A \geq 10^{-11}eV$. Furthermore, if f_A becomes smaller, the potential becomes more significant and therefore from experiment we should have had seen Axions and the fact that we did not dictates that $m_A \leq 10^{-2}$.

Moreover, Axion mass has a complicated T-dependence at $T > \Lambda_{QCD}$. Ignoring this subtlety we can estimate it :

$$\Omega_A = (\frac{10^{-6}eV}{m_a})\theta_i^2, \tag{6.10}$$

where $\theta_i = \frac{A_i}{f_A} \lesssim \pi$.



When we have $T \gg V^{1/4}$, the Axions fill the universe with random values. When Hubble parameter drops below m_A, Axions become non-relativistic particles. At this point, the

Axion starts oscillating in the harmonic like potential.

$$
\begin{aligned}
&H = \frac{T^2}{M_0}, \quad H(T_{osc}) = m_A, \quad \rho_a \sim V = m_A^2 f_A^2 \theta_i^2. \\
&\Rightarrow n_A|_i = \frac{\rho_A}{m_A} = \frac{T_{osc}^2}{M_0} f_A^2 \theta_i^2, \\
&\Rightarrow n_{A,0} = n_{a,i} \left(\frac{T_0}{T_{osc}}\right)^3 = T_0^3 \frac{f_A^2 \theta_i^2}{M_0 T_{osc}}, \\
&\Rightarrow \rho_{A,0} = m_A T_0^3 \frac{f_A^2}{M_0 T_{osc}} \theta^2 = T_0^3 \frac{\Lambda_{QCD}^4}{M_0^{3/2} m_A^{3/2}} \theta^2,
\end{aligned}
\tag{6.11}
$$

where $T_{osc} = \sqrt{M_0 m_A}$ and $f_A = \frac{\Lambda_{QCD}^4}{m_A^2}$. For $\theta \sim 1$ and $m_A \sim 10^{-6} eV$, the Axion can be dark matter.

Remark. *Note that this does not mean that in order for Axion to be dark matter it must have a very fine tuned mass. By choosing a different initial θ_i, this can be compensated.*

6.2.3 Weakly Interacting Massive Particle (WIMP)

The first thing to mention about this candidate is that it is supposed to interact through the electroweak force, though with no charge. It also has the natural production mechanism called the freeze out production mechanism. More or less, the idea is to have fermionic particles like neutrinos which are heavier and stable. The idea is to have processes where ($X + \bar{X} \leftrightarrow$ SM particles) but there are no process where X decays into standard model particles. As usual in our calculations of freeze out processes, we have :

$$
\begin{aligned}
&\Gamma_x = \langle \sigma N_x v \rangle \sim H \\
&\Rightarrow N_x = g_x \left(\frac{M_x T}{2\pi}\right)^{3/2} e^{-M_x/T}, \\
&\Rightarrow T_{freeze-out} = \frac{M_x}{\log g_x M_x M_0 \sigma_0},
\end{aligned}
\tag{6.12}
$$

where you should note that we do not have any specific formula for the cross section, but as we assume that the process is going to take place through the electroweak channel, it is of the order of the Fermi's interaction constant. Furthermore, define $\sigma_0 = \sigma v$.

$$
\begin{aligned}
&N_x(t_0) = N_x(t_f) \frac{a^3(t_f)}{a^3(t_i)} \sim N_x(t_f) \frac{T_\gamma^3}{T_f^3} \\
&\Rightarrow \Omega_x = \frac{N_x(t_0) M_x}{\rho_{critical}} \sim \frac{T_\gamma^3}{T_f^3} \frac{H_{freeze-out}}{\sigma_0} \\
&\quad = \frac{T_\gamma^3}{T_f} \frac{M_x}{M_0} = \frac{T_\gamma^3}{M_0} \frac{\log(M_0 M_x \sigma_0)}{\sigma_0} \sim 10^{-10} \frac{GeV^2}{\sigma_0} \log(M_0 M_x \sigma_0),
\end{aligned}
\tag{6.13}
$$

with $\sigma_0^{typ.} \sim 10^{-10} (\frac{E}{GeV})^2$.

6.2.4 Primordial Black Holes

It could still be that dark matter is not a new particle but is made of primordial black holes. This model was almost observationally excluded until the detection of gravitational waves which renewed some interest in the field. It is thought that there were these black holes during the inflationary perturbation era which we will discuss later and they clumped together and formed dark matter.

Remark. *Note that as we have seen during this section, dark matter can be things of very different nature! Very small fermionic particles, huge bosonic fields, primordial black holes or WIMP! The reason behind this fact is that although we have a lot of evidence from very different scales, i.e. cosmological scales, galaxy clusters, and ..., but as we do not have any laboratory experiments involving dark matter, non has been completely ruled out.*

7 Inhomogeneous Universe

Up until now, all we have done was in the homogeneous background. In this section, we will use a more realistic approach using cosmological perturbation theory. For a complete version of the derivations refer to Gorbunov and Rubakov. The reason is that of course at short distance scales $\frac{\delta\rho(\vec{x})}{\rho_0(t)} \sim 1$. Note that here by $\rho_0(t)$ we mean the homogeneous background which is independent of position and by $\delta\rho(\vec{x})$ we mean the fluctuations around this background. We will repeat these definitions when we start linearized theory. The basic idea is that perturbations grow from some "seed" or primordial perturbation where $\frac{\delta\rho(\vec{x})}{\rho_0(t)} \ll 1$. This idea is directly confirmed from recombination where $\frac{\delta\rho(\vec{x})}{\rho_0(t)} \sim 10^{-5}$. Putting all of the discussion, we see that the full treatment of perturbations requires general relativity and matter perturbations.

7.1 Jean's Instability

We first start by using Newtonian gravity and dynamics as a toy model. To further simplify the problem, we consider a non-expanding universe. We have three main equations namely the Poisson, continuity and Euler's equation :

$$\begin{aligned}
&\nabla^2\phi = 4\pi G\rho, \\
&\frac{\partial\rho}{\partial t} + \vec{\nabla}.(\rho\vec{v}) = 0 \\
&\frac{d\vec{v}}{dt} + (\vec{v}.\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}P - \vec{\nabla}\phi.
\end{aligned} \tag{7.1}$$

The first two are of course very familiar to you. We provide further motivation for the form of the Euler's equation which is basically Newton's second law.

$$\begin{aligned}
&F = ma \\
&F = F_{grav.} + F_{pressure} = -\int dV(\rho\vec{\nabla}\phi + \vec{\nabla}P) \\
&\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + \frac{\partial\vec{v}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial\vec{v}}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial\vec{v}}{\partial z}\frac{\partial z}{\partial t}.
\end{aligned} \tag{7.2}$$

The second equation is just the usual form of force in integral form. The intuition behind the last equation comes from the fact that the acceleration of the fluid is not only due to the movement of each element, but also the movement of clumps of the fluid in general. Putting these altogether, we arrive at Euler's equation. Our next step is to linearize around the homogeneous background.

$$\begin{aligned}
&\rho(\vec{x},t) = \rho_0(t) + \delta\rho(\vec{x},t), \\
&P(\vec{x},t) = P_0(t) + \delta P(\vec{x},t), \\
&\vec{v}(\vec{x},t) = \vec{v}_0(t) + \delta\vec{v}(\vec{x},t), \\
&\phi(\vec{x},t) = \phi_0(t) + \delta\phi(\vec{x},t).
\end{aligned} \tag{7.3}$$

Remark. *Remember that we are still in the non-expanding background, so we have* $H = 0, \vec{v}_0 = 0, \rho_0 = const.$ *We will see in the future that expansion will actually slow down the growth of perturbations.*

The linearized equations are :

$$
\begin{aligned}
&\nabla^2 \delta\phi = 4\pi G \delta\rho, \\
&\frac{\partial \delta\rho}{\partial t} + \vec{\nabla}.(\rho_0 \delta\vec{v}) = 0 \\
&\frac{d\delta\vec{v}}{dt} + \frac{1}{\rho_0}\vec{\nabla}\delta P + \vec{\nabla}\delta\phi = 0.
\end{aligned} \tag{7.4}
$$

Furthermore, define $v_s^2 = \frac{\delta P}{\delta \rho}$ as the speed of sound squared. Taking the time derivative of the second and the divergence of the third equation in eq.(7.4) and subtracting we get :

$$
\frac{\partial^2}{\partial t^2}\delta\rho - v_s^2 \nabla^2 \delta\rho - 4\pi G \rho_0 \delta\rho = 0. \tag{7.5}
$$

Let us first check the plane wave solutions of this equation, i.e. $\delta\rho = Ae^{i(kx-\omega t)}$.

$$
\Rightarrow \omega^2 = v_s^2 k^2 - 4\pi G \rho_0. \tag{7.6}
$$

We see that if $k < \sqrt{4\pi G}\frac{1}{v_s}$, then ω is imaginary and therefore, solutions grow or decay exponentially. The critical wavelength is :

$$
\lambda_s = \frac{2\pi v_s}{\sqrt{4\pi G \rho_0}}. \tag{7.7}
$$

If $\lambda > \lambda_s$, the solution becomes unstable. This phenomena is called **Jean's instability**.

7.2 Linearized General Relativity

First we would like to point out a few tips about what we are about to do in this subsection.

- We use the linearized Einstein equations around the FRW background (We will mostly do calculations in momentum space as it will make calculations easier just like in the Jean's instability case).
- It is important to distinguish $\lambda > r_H$ and $\lambda < r_H$ (r_H is the cosmological horizon).
- It is important to keep track of growing modes.
- We remind that prior to CMB, there are only oscillations around the homogeneous background.
- After recombination (matter domination) growth is most important and the large scale structures start to form.

Before moving on to the perturbative parameters, we review the background ingredients once more. The FRW conformal time is defined as :

$$
\begin{aligned}
&a(\eta)d\eta = dt, \\
&ds^2 = a^2(\eta)(-d\eta^2 + dx^i dx^i).
\end{aligned} \tag{7.8}
$$

From here on, we will denote differentiation w.r.t regular time with dots and w.r.t. conformal time with primes. Using this definition, we have :

$$
H = \frac{\dot{a}}{a} = \frac{a'}{a^2}. \tag{7.9}
$$

The Friedmann equations are :

$$\begin{aligned}
&\frac{a'^2}{a^4} = \frac{8\pi G}{3}\rho, \\
&\frac{2a''}{a^3} - \frac{a'^2}{a^4} = 8\pi G P, \\
&\rho' = -3\frac{a'}{a}(P+\rho).
\end{aligned} \tag{7.10}$$

Moreover, the solutions for different epochs are :

$$a(\eta) = \begin{cases} c\eta = t^{1/2} & \text{Radiation Domination} \\ c\eta^2 = t^{2/3} & \text{Matter Domination} \\ -\frac{1}{H\eta} = e^{Ht} & \text{Dark Energy Domination} \end{cases} \tag{7.11}$$

For different sectors :

$$\Omega_{DM} = 0.22, \quad \Omega_{Baryons} = 0.04, \quad \Omega_\Lambda = 0.73, \quad \Omega_{curvature} < 0.01. \tag{7.12}$$

Moreover, the matter-radiation equation happens at $z \approx 3000$. Additionally, we had :

$$\begin{aligned}
&H = H_0\sqrt{\Omega_\Lambda + \Omega_m(1+z)^3 + \Omega_{rad}(1+z)^4} \\
&\Rightarrow \eta = \int_z^\infty \frac{dz'}{a_0 H_0\sqrt{\Omega_\Lambda + \Omega_m(1+z')^3 + \Omega_{rad}(1+z')^4}}
\end{aligned} \tag{7.13}$$

Calculating numerically yields :

$$\frac{\eta_{recomb.}}{\eta_{mat.rad.eq.}} \approx 2.4, \quad \frac{\eta_0}{\eta_{mat.rad.eq.}} \approx 1.2\times 10^2, \quad \frac{\eta_0}{\eta_{recomb.}} \approx 50. \tag{7.14}$$

Now, we further define the perturbative variables in a useful manner. Note that from here on, we will use the mostly plus convention for the Minkowski metric, $\eta = diag(-1,1,1,1)$. Suppose we have :

$$ds^2 = a^2(\eta)\gamma_{\mu\nu}dx^\mu dx^\nu. \tag{7.15}$$

Therefore, $\gamma_{\mu\nu}$ is the full metric. As we are doing perturbations around the FRW metric, we write :

$$\gamma_{\mu\nu} = \eta_{\mu\nu} - h_{\mu\nu}. \tag{7.16}$$

As we are doing perturbation theory, we assume $|h_{\mu\nu}| \ll 1$.

Furthermore, the energy-momentum tensor is written as :

$$T^\mu{}_\nu = \bar{T}^\mu{}_\nu(\eta) + \delta T^\mu{}_\nu \tag{7.17}$$

Upon these definitions, the Einstein field equations and the energy-momentum conservation for perturbations are :

$$\begin{aligned}
&\delta G^\mu{}_\nu = 8\pi G\delta T^\mu{}_\nu, \\
&\delta(\nabla_\mu T^{\mu\nu}) = 0.
\end{aligned} \tag{7.18}$$

Remark. *Note that in perturbation theory, in our formulation of the metric, we raise and lower indices using the Minkowski metric η. For example :*

$$h^\mu{}_\nu = \eta^{\mu\lambda}h_{\lambda\nu}. \tag{7.19}$$

The tensor inverse to $\gamma_{\mu\nu}$ is $\gamma^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$. Therefore, we have :

$$g^{\mu\nu} = \frac{1}{a^2}(\eta^{\mu\nu} + h^{\mu\nu}). \tag{7.20}$$

In addition, we must do our gauge fixing (the gauge here refers to diffeomorphism invariance). We note that as we are doing linear perturbation theory, it is enough to restrict ourselves to linear diffeomorphisms.

$$\begin{aligned} g^{\mu\nu}(x^\rho) &\xrightarrow{\zeta} \tilde{g}^{\mu\nu}(x^\rho) = g^{\mu\nu}(x^\rho - \zeta^\rho) + \nabla^\mu \zeta^\nu + \nabla^\nu \zeta^\mu. \\ h^{\mu\nu}(x^\rho) &\xrightarrow{\zeta} \tilde{h}^{\mu\nu}(x^\rho) = h^{\mu\nu}(x^\rho) + \partial^\mu \zeta^\nu + \partial^\nu \zeta^\mu + 2\eta^{\mu\nu}\zeta^\lambda \frac{\partial_\lambda a}{a}. \end{aligned} \tag{7.21}$$

We fix three out of the four gauges for now as :

$$h_{0i} = 0. \tag{7.22}$$

Shifting our attention to the energy-momentum tensor, we will again take the ideal fluid approximation.

$$\begin{aligned} T^{\mu\nu} &= (\hat{\rho} + \hat{P})u^\mu u^\nu + \hat{P} g^{\mu\nu}, \\ \hat{\rho} &= \rho(\eta) + \delta\rho, \\ \hat{P} &= P(\eta) + \delta P. \end{aligned} \tag{7.23}$$

In our next step, we would like to use the relativity constraint on the 4-velocity, u, to express it in terms of the 3-velocity. In the rest frame, we have $\bar{u}_0 = \frac{1}{a}, \bar{u}_i = 0$.

$$\begin{aligned} & g_{\mu\nu} u^\mu u^\nu = -1 \\ \Rightarrow & u^0 = \frac{1}{a}(1 + \delta u^0), \quad u^i = \frac{1}{a} v^i \\ \Rightarrow & -(1 + h_{00})(1 + \delta u^0)^2 = -1 + \mathcal{O}(v^i v^i) \\ \Rightarrow & 2\delta u^0 = -\frac{1}{2} h_{00} \Rightarrow u^0 = \frac{1}{a}(1 - \frac{1}{2} h_{00}) \\ \Rightarrow & u_0 = -a^2(1 + h_{00}) u^0 = -a(1 + \frac{1}{2} h_{00}) \end{aligned} \tag{7.24}$$

Note that v^i is itself first order in perturbations. Notice the convenience that our gauge choice has brought us!

Remark. *Let us pause and see our results up until now. Our final result is not Lorentz invariant! This was actually expected as we are doing perturbation around a fix background. However, as we did perturbations around the FRW and chose our gauge fixing in a rotationally invariant way (because h_{0i} acts like a vector under rotations), we are still left with translational and rotational invariance.*

Now we can write the energy-momentum tensor as :

$$\begin{aligned} \delta T^0{}_0 &= \delta\rho, \\ \delta T^0{}_i &= (\rho(\eta) + P(\eta)) v_i, \\ \delta T^i{}_j &= \delta\rho \delta^i_j. \end{aligned} \tag{7.25}$$

We are finally in a position to write the energy-momentum conservation equation linearly in perturbation :

$$\begin{cases} \delta\rho' + 3\frac{3a'}{a}(\delta\rho + \delta P) + (\rho + P)(\partial_i v_i - \frac{1}{2}h') & \text{0 component} \\ \partial_i \delta\rho + (\rho + P)(4\frac{a'}{a}v_i + \frac{1}{2}\partial_i h_{00}) + (v_i(\rho + P))' = 0 & \text{i component} \end{cases} \tag{7.26}$$

where $h = h_{ii}$ is the trace of the spatial part of the $h_{\mu\nu}$ tensor.

Before we can derive linearized Einstein equations, we need to use symmetries (translational invariance), here we mean global symmetries, to go to momentum space. We will further use rotational symmetries to further simplify our calculations.

Momentum space :

$$\begin{aligned} h_{\mu\nu} &= \int d^3k e^{ikx} h_{\mu\nu}(k, \eta), \\ \delta\rho &= \int d^3k e^{ikx} \delta\rho(k, \eta), \\ \delta P &= \int d^3k e^{ikx} \delta P(k, \eta), \\ \delta v_i &= \int d^3k e^{ikx} \delta v_i(k, \eta). \end{aligned} \tag{7.27}$$

Therefore we have $\partial_i \sim ik_i$.

Remark. *Remember that the physical momentum is* $q(\eta) = \frac{k}{a(\eta)}$.

Remark. *It is important to understand why it is convenient to go to momentum space. As we are doing* ***linear*** *perturbation theory and our background metric is translational invariant, different modes (different k's) do not get mixed! That is why it is easier to do the computations in momentum space.*

Now consider some $\vec{k}$. We still have further symmetries involving rotations around $\vec{k}$. This $SO(2)$ symmetry means that different sectors (scalars, vectors and tensors) do not get mixed!

$$\begin{cases} \delta\rho, \delta P, v^i \sim k, \delta_{ij}, k_i.k_j, h = h_{ii} & \text{Scalars} \\ v_i^T : v_i^T.k_i = 0 & \text{Vectors} \\ \text{Traceless transverse part of } h_{ij} & \text{Tensors} \end{cases} \tag{7.28}$$

Therefore, we break h_{ij} as follows :

$$\begin{aligned} h_{00} &= -2\phi, \\ h_{ij} &= -2\psi\delta_{ij} - 2Ek_ik_j + i(k_iW_j^T + k_jW_i^T) + h_{ij}^T, \end{aligned} \tag{7.29}$$

where ϕ is basically the Newtonian potential. Also we have : $v_i = v_i^T + ik_iv$.

In these notes, we only derive the equations in the scalar sector as they are the most interesting ones. We also fix the only gauge left. We should be careful to chose a gauge which is consistent with our previous gauge.

$$\begin{aligned} \zeta_i = -\partial_i\sigma, &\quad \zeta_0 = \partial_\eta\sigma \\ \Rightarrow \tilde{h}_{ij} = h_{ij} &- \partial_i\partial_j\sigma - 2\frac{a'}{a}\delta_{ij}\sigma'. \end{aligned} \tag{7.30}$$

We can use this gauge to kill the E term in h_{ij} in eq. 7.29. Finally, since we are only considering the scalar part :

$$ds^2 = a^2(\eta)(-(1+2\phi)d\eta^2 + (1+2\psi)d\vec{x}^{\,2}). \tag{7.31}$$

In this gauge, the Einstein field equations (to linear order) in the scalar sector are :

$$\begin{aligned}
&\psi = -\phi, \\
&\nabla^2\phi - 3\frac{a'}{a}\phi' - 3\frac{a'^2}{a^2}\phi = 2\pi G a^2 \delta\rho_{tot}, \\
&\phi' + \frac{a'}{a}\phi = -4\pi G a^2[(\rho + P)v]_{tot}, \\
&\phi'' + 3\frac{a'}{a}\phi' + 2\left(\frac{a''}{a} - \frac{a'^2}{a^2}\right)\phi = 4\pi G a^2 \delta\rho_{tot},
\end{aligned} \tag{7.32}$$

where the tot index means that all different matter components should be considered such as DM, DE, and radiation. We also have the energy-momentum conservation. We note that as we do not consider the interaction between the different matter components, these equations can be considered for each matter component individually :

$$\begin{aligned}
&\delta\rho'_\lambda + 3\frac{a'}{a}(\delta\rho_\lambda + \delta P_\lambda) + (\rho_\lambda + P_\lambda)(\nabla^2 v - 3\phi') = 0 \\
&((\rho_\lambda + P_\lambda)v_\lambda)' + 4\frac{a'}{a}(\rho_\lambda + P_\lambda)v_\lambda + \delta\rho_\lambda + (\rho_\lambda + P_\lambda)\phi = 0.
\end{aligned} \tag{7.33}$$

Of course these equations follow from the Einstein equations but they are useful in this form, that is why we stated them as well. Our final equation is the equation of state :

$$\delta P_\lambda = u_{s\lambda}^2 \delta\rho_\lambda, \tag{7.34}$$

where $u_{s\lambda}$ is the speed of sound of the λ matter component.

Subhorizon-Superhorizon modes

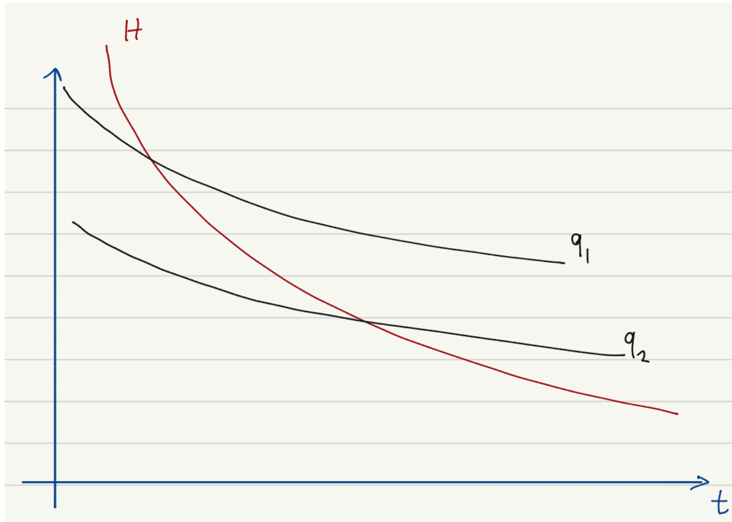
We open a subject in this part that we shall use its terminology in the following sections. We call a particular mode $\vec{k}$ a superhorizon mode if we have:

$$q(\eta) = \frac{k}{a(\eta)} \ll H(\eta). \tag{7.35}$$

Similarly, we call a particular mode $\vec{k}$ a subhorizon mode if we have :

$$q(\eta) = \frac{k}{a(\eta)} \gg H(\eta). \tag{7.36}$$

Note that the subhorizon and superhorizon refers to the wavelength, e.g. a small $\vec{k}$ has large wavelength. Now if we consider a decelerating FRW universe $a < ct$, e.g. $a \sim t^{1/2}$ or $t^{2/3}$, whereas $H \sim \frac{\dot{a}}{a} \sim \frac{1}{t}$. Therefore, $\frac{q}{H}$ grows as time evolves. Therefore, modes enter the horizon (see the figure below). Of course by the same logic, the opposite happens for an accelerating universe.



Back to Perturbations

Furthermore, by subtracting the Einstein field equations we will have :

$$\phi'' + 3\frac{a'}{a}\phi'(1+u_s^2) + [2\frac{a''}{a} - \frac{a'^2}{a^2}(1-3u_s^2)]\phi + u_s^2 k^2 \phi = 0. \tag{7.37}$$

Note that the above equation looks like a wave equation (the first and last term) alongside a friction term (the second term) due to the expansion of the universe. Moreover, define $u_s H^{-1}$ as the sound horizon (the distance travelled by sound since the Big Bang).

Furthermore, if we restrict ourselves to only a single component fluid, we know that in the background we have $P = \omega\rho$ and $\omega = u_s^2$. Using Friedmann equations we have :

$$-8\pi G a^2 (P - u_s^2 \rho) = 0. \tag{7.38}$$

Therefore, the equation further simplifies to :

$$\phi'' + 3\frac{a'}{a}(1+u_s^2)\phi' + u_s^2 k^2 \phi = 0. \tag{7.39}$$

Now if we take $\lambda \ll u_s H^{-1}$:

$$\begin{aligned} &\phi'' + 3\frac{a'}{a}(1+u_s^2)\phi' = 0 \\ \Rightarrow &\phi'' + \frac{c}{\eta}\phi' = 0. \end{aligned} \tag{7.40}$$

Therefore, there are two different solutions.

$$\phi = \text{const}, \quad \phi = \eta^{1-c}. \tag{7.41}$$

Note that as $c > 1$, the non-constant solution dies as $\eta \to \infty$. That is why we only consider constant answers to be of importance. Therefore, the modes are all constant before they enter. We can also see that longer modes enter the horizon later in time.

Let us focus on density perturbations. Define $\delta = \frac{\delta\rho}{\rho}$. Note that this parameter is easily accessible through observations. We have :

$$-3\frac{a'^2}{a^4}\phi = 4\pi G a^2 \delta\rho. \tag{7.42}$$

Again using Friedmann equation : $\frac{a'^2}{a^4} = \frac{8\pi G}{3}\rho$, we get δ =const.

Relativistic Matter

In radiation domination we have $a \sim \eta$ and for relativistic matter we have $\omega = u^2 = \frac{1}{3}$.

$$\Rightarrow \phi'' + \frac{4}{\eta}\phi' + u_s^2 k^2 \phi = 0. \tag{7.43}$$

Motivated by what we found in the previous part, we take the following as our initial condition. Therefore, the solution is :

$$\phi(\eta) = -3\phi_i \frac{1}{(u_s k\eta)^2}\left[\cos(u_s k\eta) - \frac{\sin(u_s k\eta)}{u_s k\eta}\right] \tag{7.44}$$

We see that in the $k\eta \to 0$ limit, we get ϕ_i and in the $k\eta \to \infty$ limit we get $\frac{\phi_i(k)\cos(u_s k\eta)}{(u_s k\eta)^2}$. Now at lagre k, we can use the Einstein equations eq.7.32 and only take the dominating term $(k^2\phi)$:

$$\begin{aligned}\delta\rho &= -\frac{1}{4\pi G}\frac{k^2\phi}{a^2}\\ \Rightarrow \delta_{rad} &= \frac{\delta\rho_{rad}}{\rho_{rad}} = 6\phi_i\cos(u_s k\eta),\end{aligned} \tag{7.45}$$

which is an oscillatory solution. This means that theere is no Jean's instability in this case.

Remark. *Note that if you do the same calculations for the subhorizon modes in radiation domination for non-relativistic matter, you will see a logarithmic growth.*

Non-relativistic Matter

In this case, we have :

$$\phi'' + 3\frac{a'}{a}(1 + u_s^2)\phi' + u_s^2 k^2 \phi = 0, \tag{7.46}$$

but as $u_s = 0$, we get :

$$\phi'' + 3\frac{a'}{a}\phi' = 0. \tag{7.47}$$

Note that in this case modes are always superhorizon as the term which could dominate for $k \to \infty$ is always zero! So we get a constant and a decaying mode but again we only need to consider the constant mode. Again using the Einstein equation :

$$\delta\rho = -\frac{1}{4\pi G a^2}(k^2 + \frac{12}{\eta^2})\phi. \tag{7.48}$$

Furthermore, remember that we have $\rho \sim \frac{1}{a^3}$.

$$\Rightarrow \delta_{matter}^{subhorizon} \sim a \Rightarrow \text{ Jean's Instability!} \tag{7.49}$$

Remark. *Note that here by subhorizon we mean the cosmological horizon as there is no notion of sound horizon!*

Remark. *note that when the other term dominates, it is confirmation of our previous result that δ is constant!*

The linear growth that we found here is the seed of all the structure that we see in the universe. This computation is most relevant for dark matter modes that enter the horizon at or after matter-radiation equality.

Up until now we did not talk about the size of ϕ. Many different observation lead to $\phi_i \sim 3 \times 10^{-5}$ and as we have $\frac{a_0}{a_{eq}} \sim 3000$, we get $\delta \sim 0.09$ and if done more carefully, you get $\delta \sim 0.03$. Now we should ask, what type of mode are these?

$$k\eta_{eq.} = 1, \quad q = \frac{k_{eq.}}{a_0} = \frac{1}{a_0 \eta_{eq.}}. \tag{7.50}$$

We know that $a_0\eta_0$ is roughly the size of the universe, 1.4×10^4 Mpc.

$$\Rightarrow \lambda = \frac{2\pi}{q} \sim 750\text{Mpc}. \tag{7.51}$$

Therefore the universe is still dominant linearly on those scales!

If we now move on to multi-component fluid, we get what is called the adiabatic mode and the Isocurvature modes. The adiabatic mode is exactly what we saw in the single component case which originate from some ϕ_i. However, in the single component case, the adiabatic mode was forced upon us by Einstein equations. In the multi-component instance, it could have been the case that there was no initial ϕ_i and the different $\delta\rho_\lambda$s created the deviations. It turns out that even in the multi-component version, we only have the adiabatic modes! The isocurvature modes have been bounded by experiment to be below 1 percent.

Remark. *Note that the fact that we name adiabatic modes as modes which start from some ϕ_i is a gauge dependent statement. This means that this is only true in our conformal Newtonian gauge.*

7.3 CMB Fluctuations

This is seemingly the most complicated part of these lectures. It is told to be one of the greatest predictions of theoretical physics! It was observed in experiment in the last few decades. We sketch the computations of this part. You can find a more detailed computation in chapter 9 of Gorbunov and Rubakov book.

Remember that baryons and photons were tightly coupled before recombination. Furthermore, their perturbations follow :

$$\delta = 6\phi_i(k) \cos(u_s k\eta). \tag{7.52}$$

At recombination, photons decouple and from then on they fly towards us. Basically, they are free electromagnetic waves which are only impacted by the expansion of the universe.

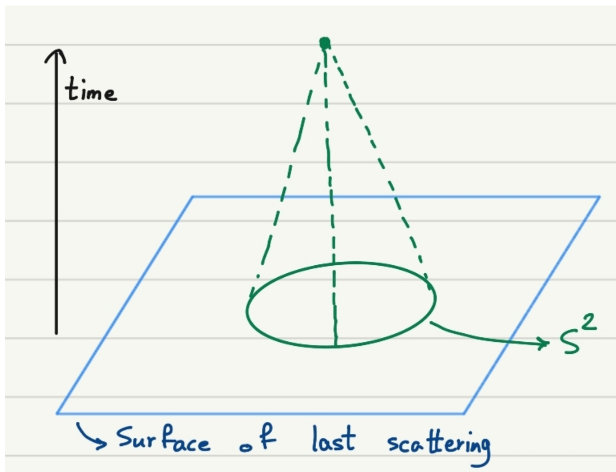
Remark. *Note that even at this point we are not being perfectly rigorous as the gravitational potentials alter the way the photons are redshifted but we can ignore this subtlety for now.*

Additionally, photons coming from different angles have different temperatures.

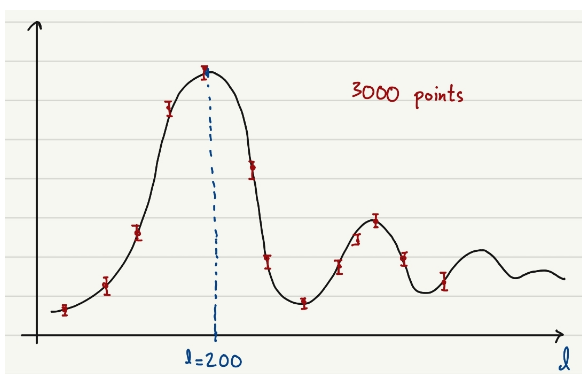
$$\frac{\delta T_\gamma}{T_\gamma}(\theta) \sim \frac{1}{4}\delta_{B\gamma} + \Phi + \dots \tag{7.53}$$

Where dots stand for other effects like Doppler shift, integrated Sachs-Wolfe effect, which turn out to be subleading at the scales of the first few CMB peaks that we will focus on.

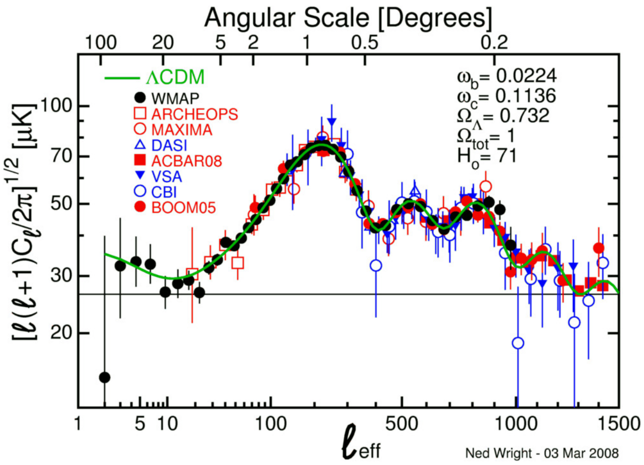
This is the picture to have in mind where our observations happen at the point in the top.



It turns out that we can predict these fluctuations in momentum space and it looks something like this :



The actual plot is this [2]:



Basically, the bumps you see in the above plots are consequences of eq.7.52. The $\phi_i(k)$ are called the primordial spectrum which we have not yet talked about (the initial conditions).

At $\eta = \eta_r$, some k's are excited and some are not. We have:

$$\begin{aligned}
&\int e^{ikx(\theta)} \sin(2\pi l\theta) d\theta \\
\Rightarrow &\text{Peaks}: \; u_s k_n \eta_r = n\pi \Rightarrow k = \frac{n\pi}{u_s \eta_r} \\
\Rightarrow &\Delta x_n = \frac{2\pi}{k_n} = \frac{2u_s\eta_r}{n} \Rightarrow \Delta\theta = \frac{\Delta x}{\eta_0 - \eta_r} \sim \frac{\Delta x}{\eta_0}, \quad l = \frac{2\pi}{\Delta\theta} \\
\Rightarrow &l = n\pi\sqrt{3}\frac{\eta_0}{\eta_r} \approx 300n.
\end{aligned} \tag{7.54}$$

We can see from the data that the difference between consecutive peaks are actually 300 as well which agrees with the above computations. Had we wanted to get a more precise result, we could have considered the Legendre polynomials and acquired a more accurate description.

For Gaussian random fields we have :

$$\langle \delta_{k_1} \delta_{k_2} \rangle = \delta(\vec{k_1} - \vec{k_2}) P(|\vec{k}|). \tag{7.55}$$

These are random but correlated (average over modes with the same $|\vec{k}|$ and different angles) and they produce the plot sensitive to $P(|\vec{k}|)$. It is also sensitive to $\Omega_m, \Omega_B, \Omega_\Lambda$ and Ω_c. So far, we have discussed evolution of initial conditions from Big Bang singularity until the period when they can be observed. Result of observations show that :

$$P(|\vec{k}|) = 10^{-5} \left(\frac{k_*}{k}\right)^{3+(n_s-1)}, \tag{7.56}$$

where $n_s \approx 0.96$. This shows a couple of points :

- perturbations are small and approximately Gaussian.
- They are correlated on superhorizon scales.
- They have a relatively simple power spectrum.

Moreover, this does not look like a generic "Big Bang" and suggests to look for a simple theory that proceeds the Big Bang. The most popular candidate is **Inflation**.

8 Inflation

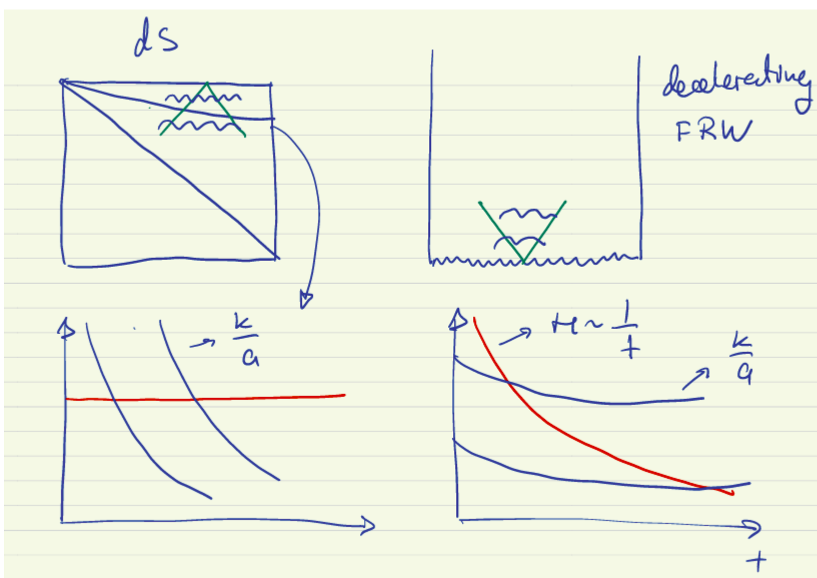
Inflation is a theory for initial conditions that naturally explains the P (power spectrum) that is observed. It is a period of quasi-exponential expansion before the Big Bang. There are numerous microscopic realizations of inflation that lead to similar observations. We will discuss a particular model called single field slow-roll inflation, first in the homogeneous background and then in perturbations. Finally we will show that quantum mechanics creates classical perturbations that naturally have the required properties.

8.1 Recap of de-Sitter space

For de-Sitter space, we have a cosmological constant, $\omega = -1$. The flat slicing is where $\rho = const, a = e^{Ht}$ where $H = \sqrt{\frac{\Lambda}{3}}$. What is important is that there is no singularity at $t = -\infty$. However, the full spacetime is obtained using closed slicing :

$$\begin{aligned} &\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \frac{\Lambda}{3} = 0 \\ &2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \Lambda = 0 \\ &\Rightarrow a = H^{-1}\cosh(Ht). \end{aligned} \tag{8.1}$$

We are interested in the expanding part of the de-Sitter space.

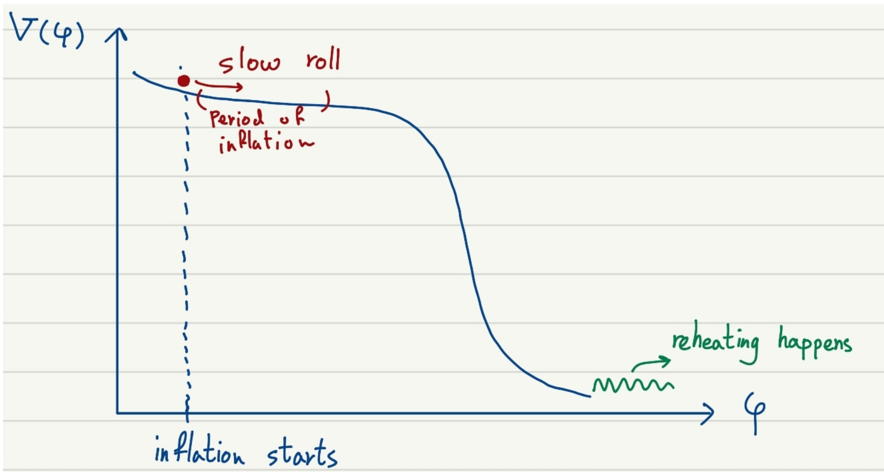


However, the problem is that if space is produced only by a rigid cosmological constant, it lasts forever and matter or radiation will never dominate! Remember that :

$$H = H_0\sqrt{\Omega_\Lambda + \Omega_m(1+z)^3 + \Omega_rad(1+z)^4}. \tag{8.2}$$

8.2 Slow-roll Inflation

The solution is to consider inflation as Λ + Clock. The clock will tell when Λ decays. This leads to some time dependent Λ. This is what is called the quasi de-Sitter space. Single field slow-roll inflation is one realization of this model:



Reheating is where the energy $\sim \Lambda$ gets transmitted into radiation and matter, $T^4 \sim \Lambda$. We must note that the potential does not have to have this particular form, but it is good to imagine it for simplicity. For a single scalar field we have:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{8.3}$$

The energy-momentum would be :

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}. \tag{8.4}$$

We first use an approximation close to de-Sitter :

$$g_{\mu\nu} = \text{diag}(-1, a^2, a^2, a^2), \quad a = e^{Ht}. \tag{8.5}$$

Therefore, in the homogeneous approximation, $\phi = \phi(t)$, we have :

$$\begin{aligned} &\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \\ &\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ &P = \frac{1}{2}\dot{\phi}^2 - V(\phi) \end{aligned} \tag{8.6}$$

Furthermore, we have the Friedmann equation :

$$H^2 = \frac{8\pi}{3M_{pl}^2}(\frac{1}{2}\dot{\phi} + V(\phi)). \tag{8.7}$$

The slow-roll regime is when :

$$\left|\frac{\ddot{\phi}}{3H\dot{\phi}}\right| \ll 1. \tag{8.8}$$

This makes intuitive sense if you think of it in terms of friction and acceleration. Furthermore, if we consider the space to be close to de-Sitter, H is roughly constant, we then have $P \approx \rho$ which means:

$$|\frac{\dot{\phi}}{2V(\phi)}| \ll 1. \tag{8.9}$$

We can massage this approximation and express them as a condition on the potential. Using eq.8.8 we can write :

$$\dot{\phi} = -\frac{1}{3H}V'(\phi). \tag{8.10}$$

If we take the close to de-Sitter condition, eq.8.9, we have :

$$H = \frac{1}{M_{pl}}\left(\frac{8\pi V}{3}\right)^{1/2}. \tag{8.11}$$

This means that the potential is basically a cosmological constant. If we substitute eq.8.10 in the near de-Sitter condition we have :

$$\epsilon \equiv \frac{M_{pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2 \ll 1. \tag{8.12}$$

ϵ is called the first slow-roll parameter. On the other hand, the other slow-roll condition can also be expressed in terms of the potential using eq.8.10:

$$\begin{aligned} \ddot{\phi} &= M_{pl}\left(\frac{V''}{\sqrt{V}} - \frac{1}{2}\frac{V'^2}{V^{3/2}}\right)\dot{\phi} \\ &\sim M_{pl}^2\left(\frac{V''}{V} - \frac{1}{2}\frac{V'^2}{V^2}\right)H\dot{\phi} \end{aligned} \tag{8.13}$$

The second term on the right hand side is negligible due to the condition on ϵ and therefore we have:

$$\eta = \frac{M_{pl}^2}{8\pi}\frac{V''}{V} \ll 1. \tag{8.14}$$

ϵ and η are called the slow-roll parameters. Let us take some example. We will see that the simplest potential, the harmonic potential, can actually satisfy the slow-roll condition! Consider:

$$V(\phi) = \frac{1}{2}m^2\phi^2. \tag{8.15}$$

Both of the slow-roll conditions yield $\frac{M_{pl}^2}{\phi^2} \ll 1$. Furthermore, note that in order to be able to do perturbation theory, we must have $m^2\phi^2 \ll M_{pl}^4$. So the quadratic potential works as an example of inflation in this regime. Basically we want the field to be very large and the mass to be very small. We just want to emphasize that it is not so hard to find potentials which satisfy this condition. Note that by now the purely quadratic potential has been excluded by observations of the CMB.

8.3 Perturbations

The first thing to review is the free massless scalar field in Minkowski space.

$$S = -\frac{1}{2}\int dx^4 \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi,$$
$$E = H = \frac{1}{2}\int dx^3 \left[\dot{\phi}^2 + (\partial_i\phi)^2\right]. \quad (8.16)$$

We can take this Hamiltonian and recognize that in momentum space different modes separate and we are left with decoupled harmonic oscillators:

$$\phi(\vec{x},t) \xrightarrow[Fourier]{e^{i\vec{q}.\vec{x}}} \phi(\vec{q},t) \to \hat{\phi}(\vec{q},t) \quad (8.17)$$

$$\Rightarrow \hat{\phi}(\vec{x},t) = \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega}}\left(e^{i\omega t - i\vec{q}.\vec{x}}A_q^\dagger + e^{-i\omega t + i\vec{q}.\vec{x}}A_q\right), \quad (8.18)$$

where $|\omega_q| = |\vec{q}|$, $A_q^\dagger$ and A_q are creation and annihilation operators.

$$\left[A_q, A_{q'}^\dagger\right] = \delta^3(q-q') \quad (8.19)$$

Using the above to calculate the Hamiltonian, we get:

$$H = \int d^3q\omega_q A_q^\dagger A_q. \quad (8.20)$$

Remark. *Remember the single harmonic oscillator.*

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \quad (8.21)$$

We had $x = \frac{a+a^\dagger}{\sqrt{\omega}}$ *and:*

$$a = \sqrt{\frac{m\omega}{2}}(X + \frac{iP}{mw})$$
$$a^\dagger = \sqrt{\frac{m\omega}{2}}(X - \frac{iP}{mw}) \quad (8.22)$$

We get the Hamiltonian to be $H = \omega a^\dagger a$. *Moreover, in the ground state* $\langle X\rangle = 0$ *and* $\langle X^2\rangle = \frac{1}{m\omega}$.

We prepare ourselves for going from x to $\phi(\vec{q}) \to \phi_i$. The direct analog of the harmonic oscillator is the following.

$$\Rightarrow \langle\phi_q\phi_{-q'}\rangle = \delta^3(\vec{q}-\vec{q'})\frac{1}{\omega_q} = \delta^3(\vec{q}-\vec{q'})\frac{1}{|q|}. \quad (8.23)$$

Now we get to inflationary perturbations (For further reading refer to part 13.1 of Rubakov's book or arXiv 16.09.00716).

The full procedure is in principle the following. At first we need to repeat the steps we did at the classical level for radiation and matter domination with some matter sources for inflationary theory (fix the gauge, the scalar vector tensor decomposition, ...). In the next step, we need to quantize all those fields and not just the scalars. We basically need to quantize gravity as even if we only care about the scalar modes, we remember that there are scalars which come from the metric, e.g. gravitational potential. So if one wants to do inflation, one is forced to do quantum gravity.

We will do a simplified treatment. We just quantize the perturbations of the Inflaton field. We ignore that the metric is dynamical in the sense that the metric perturbation is frozen. Therefore, we have a fixed $g_{\mu\nu}$ which is just de-Sitter space. One might argue that in the classical theory, the dominant contribution came from the gravitational potential. It turns out that for slow-roll inflation, this is still a good approximation. This is not trivial but in first order of ϵ and η it gives the same answer. Basically the effects that we ignore when we fix the metric to be rigid are subleading correction in these parameters. Note that there is nothing conceptually complicated to include metric perturbations but technically it is hard and that is why we will not do it.

Inflaton Field Perturbations

Note that we change our notation but this is the accepted notation in the community. We call the Inflaton field $\phi(\vec{x},t)$. Try not to confuse it with the gravitational potential field! Define:

$$\phi(\vec{x},t) = \phi_{cl}(t) + \varphi(\vec{x},t). \tag{8.24}$$

The action for the perturbative part is:

$$S_\varphi = \frac{1}{2}\int d^4x\sqrt{-g}(-g_{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V''(\phi_{cl})\varphi^2) \tag{8.25}$$

What we did is that we took the full action of the scalar field and expanded it around the classical path. However, we can ignore the second term in the action as it is proportional to $\eta \ll 1$. Note that quadratic actions give rise to linear equations of motion. This means that what we did corresponds to linearizing the equation of motion. The equation of motion is (in FRW time):

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{1}{a^2}\Delta\varphi = 0. \tag{8.26}$$

In conformal time, it is:

$$\varphi'' + 2\frac{a'}{a}\varphi' - \Delta\varphi = 0. \tag{8.27}$$

It is in fact not so different to do calculations at the linear quantum or classical systems as you will see. Note that the second term in the above equation is of order $\frac{1}{|\eta|}k$ whereas the last term is of order k^2. Just like before, there are two regimes (the reason we put an absolute value is that η is negative for de-Sitter space).

- If $\frac{k}{|\eta|} \gg k^2 \Rightarrow |\eta k| \ll 1$ or Horizon scale then the mode is outside horizon.
- If $\frac{k}{|\eta|} \ll k^2 \Rightarrow |\eta k| \gg 1$ or Horizon scale then the mode is inside horizon.

This is a manifestation of the fact that modes exit the horizon. When time goes on, η decreases and we transition from inside horizon to outside horizon. This means that if we take time going to minus infinity for some fixed k, which means $|\eta| \to \infty$, our equations simply becomes:

$$\varphi'' - \Delta\varphi = 0. \tag{8.28}$$

The same as flat space. This is why we expect ground state oscillations.

In addition, define $\chi \equiv a(\eta)\varphi$. If we express the action in terms of χ:

$$\begin{aligned} S_\chi &= \frac{1}{2}\int d^3x d\eta \left(\chi'^2 - (\partial_i\chi)^2 + \frac{a''}{a}\chi^2\right) \\ &= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3} d\eta \left(\chi'(\vec{k})\chi'(-\vec{k}) - (\vec{k}^2 + \frac{a''}{a})\chi(\vec{k})\chi(-\vec{k})\right). \end{aligned} \tag{8.29}$$

You see that this is a harmonic oscillator where the frequency is time dependent. We can again decompose the χ field:

$$\chi(\vec{x},\eta) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2k}}\left(e^{-i\vec{k}.\vec{x}}\chi^+(\eta)A_k^\dagger + e^{i\vec{k}.\vec{x}}\chi^-(\eta)A_k\right) \tag{8.30}$$

One might ask what χ^+ and χ^- are. In flat space we had $e^{\pm\omega_k\eta}$ and $\omega_k = |\vec{k}|$. Remember that we had these exponents as they were solutions of the equations of motion ($\ddot{\varphi}_k = \omega_k\varphi_k$). This comes from the fact that quantum operators satisfy the same equations of motion as classical fields:

$$\dot{p} = [H,x] = \omega^2 x, \quad \dot{x} = [H,p] = p. \tag{8.31}$$

Now for the classical part we have:

$$\begin{aligned} &\chi_v''^{cl} - \frac{2}{\eta^2}\chi_k^{cl} + k^2\chi_k^{cl} = 0 \\ \Rightarrow &\chi^\pm = e^{\pm ik\eta}\left(1 \pm \frac{i}{k\eta}\right). \end{aligned} \tag{8.32}$$

Note that the initial conditions are fixed by matching to flat space vacuum fluctuations at $|k\eta| \gg 1$.

However, at late times we have:

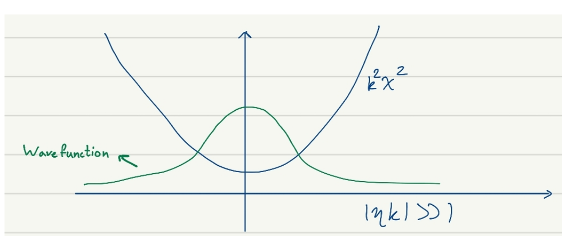
$$\langle\varphi(k)\varphi(k')\rangle \sim \delta^3(k+k')a^{-2}(\eta)\frac{1}{k}.\frac{1}{|k\eta|^2} = \delta^3(k+k')\frac{H^2}{k^3}. \tag{8.33}$$

Note that the first factor of $\frac{1}{k}$ was present in Minkowski space as well but the other two are new. You see that $\frac{H^2}{k^3}$ is the same scale invariant power spectrum we saw in CMB anisotropies. We will also see why it is not exactly 3 in inflation. Note that we call this scale invariant as $d^3k\frac{1}{k^3}$ is invariant. This is just the fluctuations of the Inflaton field. Now how do we go to the gravitational potential?

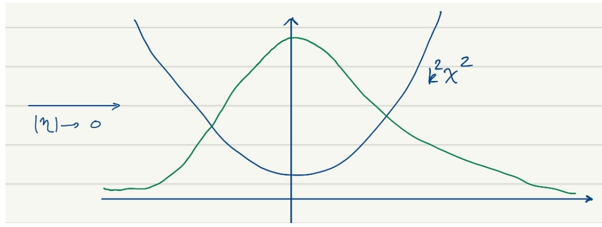
First observe that these superhorizon modes are (semi-)classical. The condition for a state to be semi-classical is:

$$\langle n| X^2 |n\rangle \gg \langle 0| X^2 |0\rangle = \frac{1}{\omega} = \frac{1}{k}. \tag{8.34}$$

Basically what happens is that at early times we have the wavefunction at its ground state:



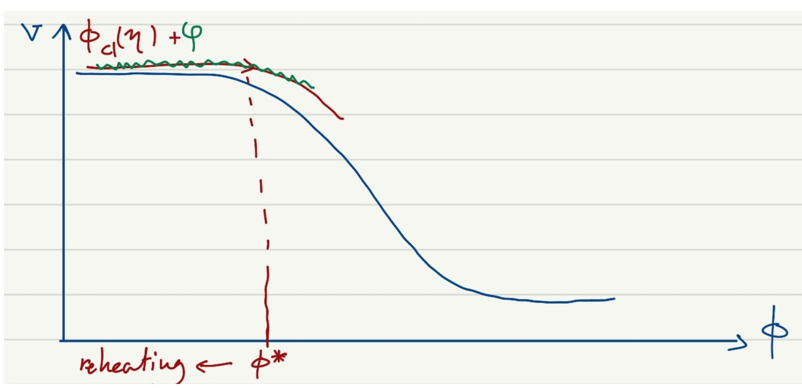
However, as time goes on, we go to the highly excited levels of the harmonic oscillator:



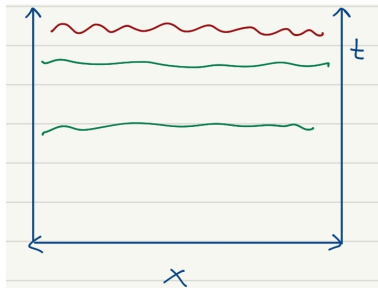
In essence, having this time dependent frequency keeps creating this quanta until it becomes a classical state! This process enables a transition to a classical description for superhorizon modes by the end of inflation. It's important to note that these superhorizon modes are the primary focus of our interest (see Figure 1).

Figure 1: Evolution of physical Modes in Inflationary Cosmology

During inflation, modes exit the horizon, and later, they re-enter the horizon. It is after this re-entry that we observe them. So what does this classical description mean?



The perturbations, φ, on top of the solution, create these oscillations (green lines in the figure). This is triggered by this quantum perturbations that are at first quantum mechanical and as the whole universe expands they become classical. This means that now if we look how this happens in spacetime, lines of constant φ are not straight lines exactly because of the quantum fluctuations. There is some critical value ϕ^*, where reheating happens, that is the point our slow-roll approximation breaks down.



You see that it happens at different moments of time at different places in the universe due to the fluctuations. Now, we can say that the reheating time is itself a function of coordinates (note that we can say this in a fixed gauge but this is the most intuitive picture).

$$\delta t_{reheating}(\vec{x}) = \frac{\varphi(\vec{x})}{\dot{\phi}_{cl}} \tag{8.35}$$

We proceed to calculating the gravitational potential:

$$\text{gravitation potential } \phi_i = \frac{\delta a}{a}\bigg|_{t_{reheating}} = \frac{\dot{a}\frac{\varphi}{\dot{\phi}_{cl}}}{a} = \frac{H\varphi}{\dot{\phi}_{cl}} \tag{8.36}$$

Moreover, we know that $\dot{\phi}_{cl} = \sqrt{V\epsilon} = HM_{pl}\sqrt{\epsilon}$.

$$\Rightarrow \phi_i = \frac{H\varphi}{HM_{pl}\sqrt{\epsilon}} = \frac{\varphi}{M_{pl}\sqrt{\epsilon}}, \tag{8.37}$$

where ϕ_i is now a classical random variable which is the enhanced quantum mechanical oscillation.

$$\begin{aligned} \langle\phi_i\phi_i\rangle_{\text{width of distribution}} &= \frac{1}{M_{pl}^2\epsilon}\langle\varphi\varphi\rangle \\ &= \frac{H^2}{\epsilon M_{pl}^2}\frac{1}{k^3}\delta^3(\vec{k}-\vec{k}') \end{aligned} \tag{8.38}$$

To match observations, we must have:

$$\frac{H^2}{\epsilon M_{pl}^2} \approx 10^{-10}. \tag{8.39}$$

In our approximation which was exact de-Sitter space, we obtain $\frac{1}{k^3}$ but if we include slow-roll corrections, one gets a power spectrum that is:

$$P(k) = \frac{1}{k^{3+(1-n_s)}}. \tag{8.40}$$

For our simple model: $n_s - 1 = 2\eta - 6\epsilon$. Experimentally, $n_s = 0.965 \pm 0.004$ which is consistent with $\epsilon, \eta \sim 10^{-2}$ which means our slow-roll approximation is justified.

References

[1] Sean M. Carroll. *Spacetime and Geometry: An Introduction to General Relativity*. Cambridge University Press, 2019.

[2] *Cosmic Microwave Background Anisotropy*. `https://astro.ucla.edu/~wright/CMB-DT.html`. Accessed: 2024-05-20.